Partition games, compositionally

Towards game comonads for linear algebraic logics

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Talk outline

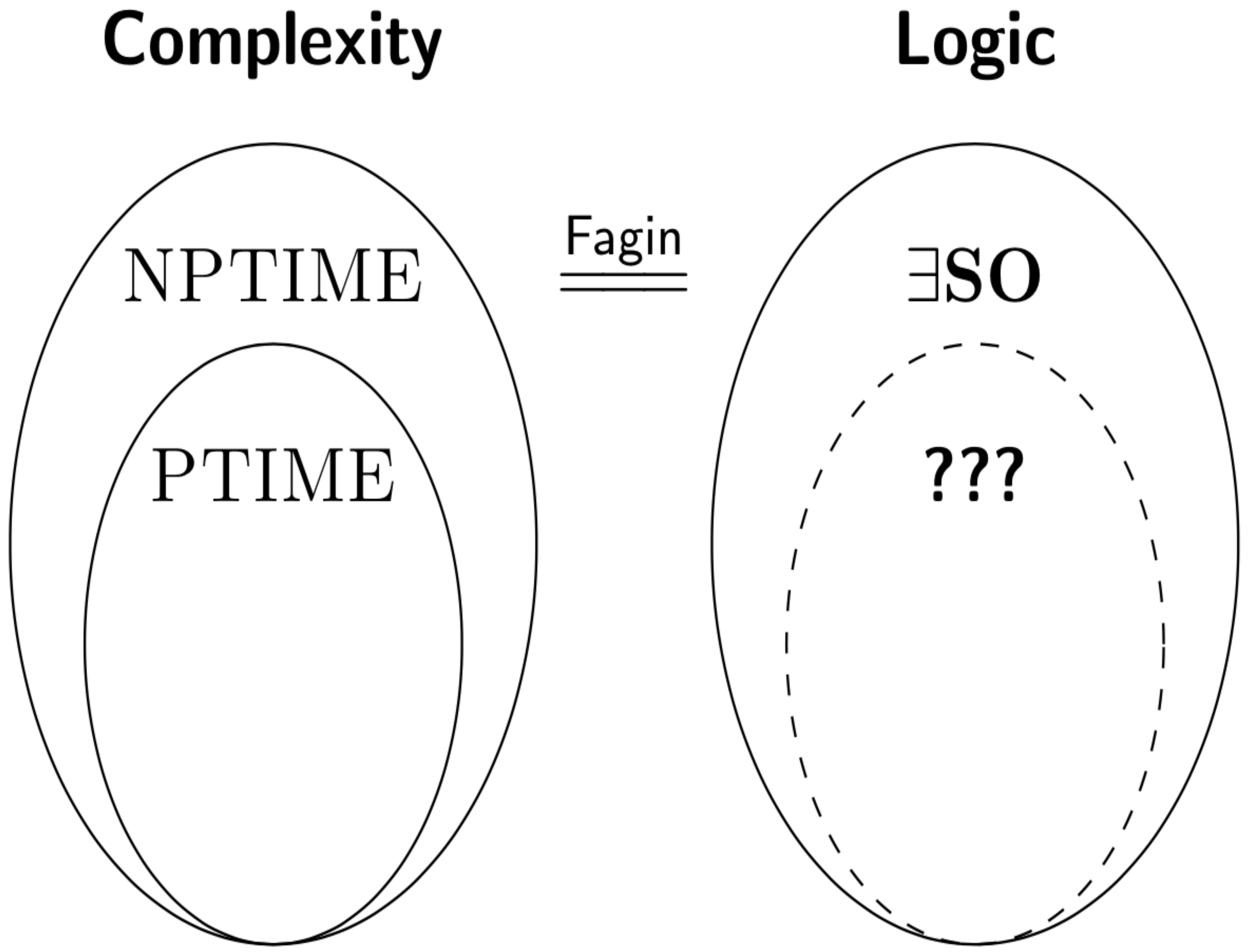
- Games in logic, finite model theory and descriptive complexity
- Game comonads so far: a powerful categorical semantics for logic games
- Partition games & their relation to linear algebraic logic
- Obstacles, progress and open questions in relating linear algebra and game comonads

Games in Descriptive Complexity & Finite Model Theory
A crash course

Descriptive Complexity

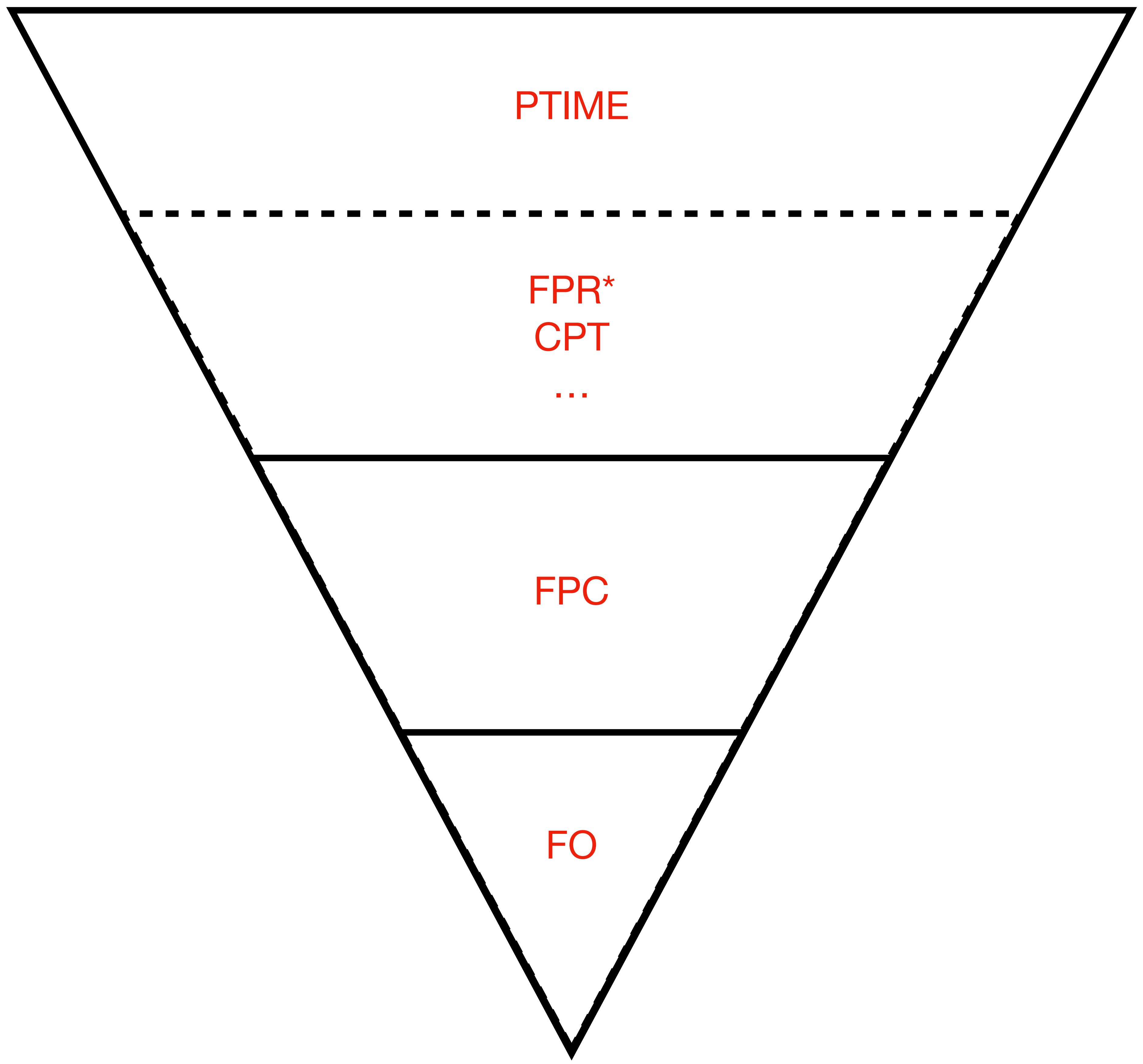
A quick tour

- (Fagin's Theorem, 1973)
  A class of finite structures is decidable in NP if and only if it is expressible in $\exists$SO
- (Gurevich's Conjecture, 1988)
  There is no equivalent logic for P
- Candidate logics for P include rank logic, and choiceless polynomial time.



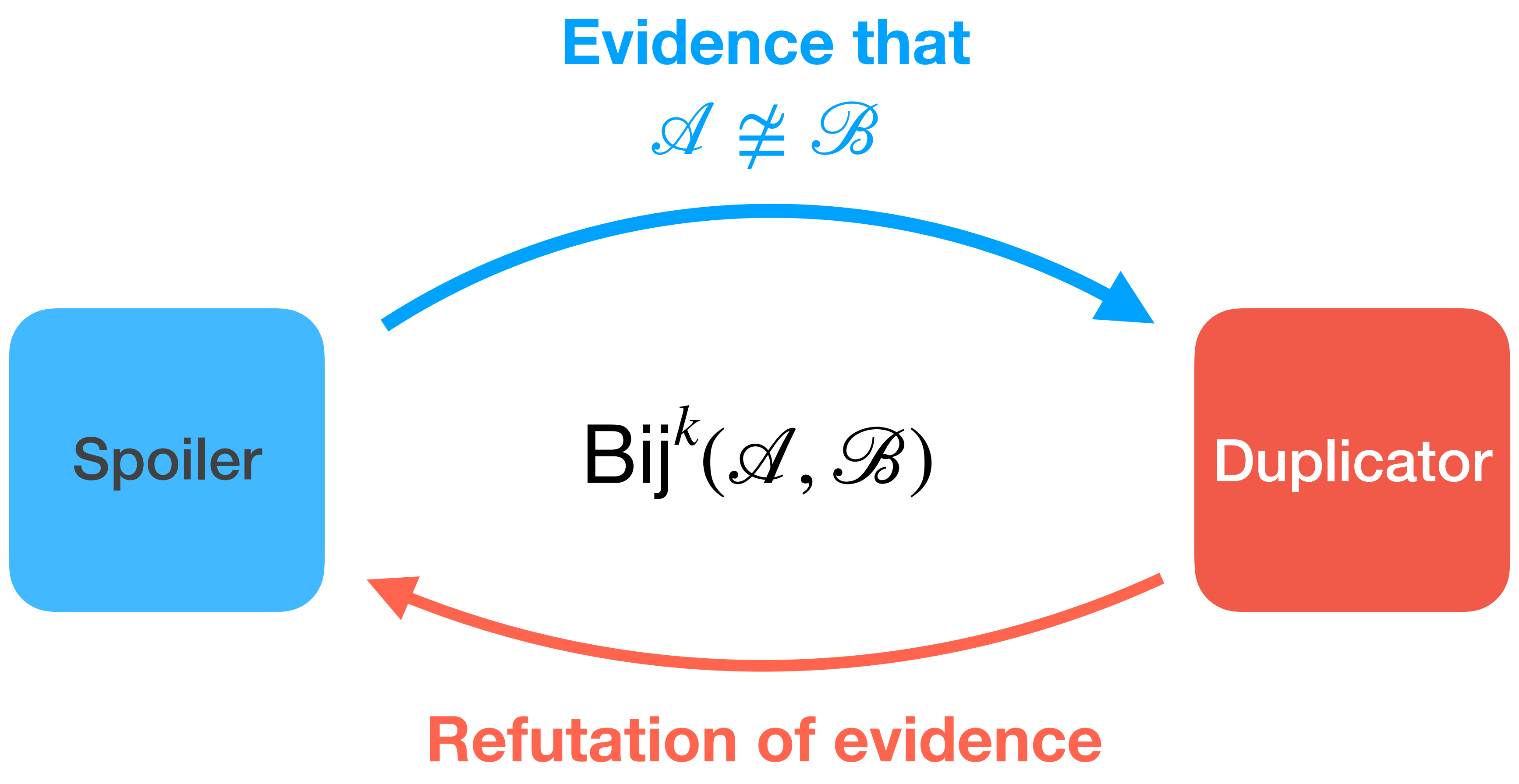
The hunt for a logic for PTIME

- FO can't even express parity or connectedness.
- FPC captures PTIME on totally ordered finites structures. (Immerman, Vardi)
- FPC does not capture P on all structures (Cai, Furer Immerman, 1992)
- Other logics have been suggested which extend the power of FPC.



Spoiler-Duplicator games used to prove upper bounds

- Expressiveness upper bounds :
  $\{\mathscr{A}_k\}$ all with P, $\{\mathscr{B}_k\}$ all lacking P
  Show that $\mathscr{A}_k \equiv_{\mathscr{L}_k} \mathscr{B}_k$ then P inexpressible in $\mathscr{L} = \cup \mathscr{L}_k$



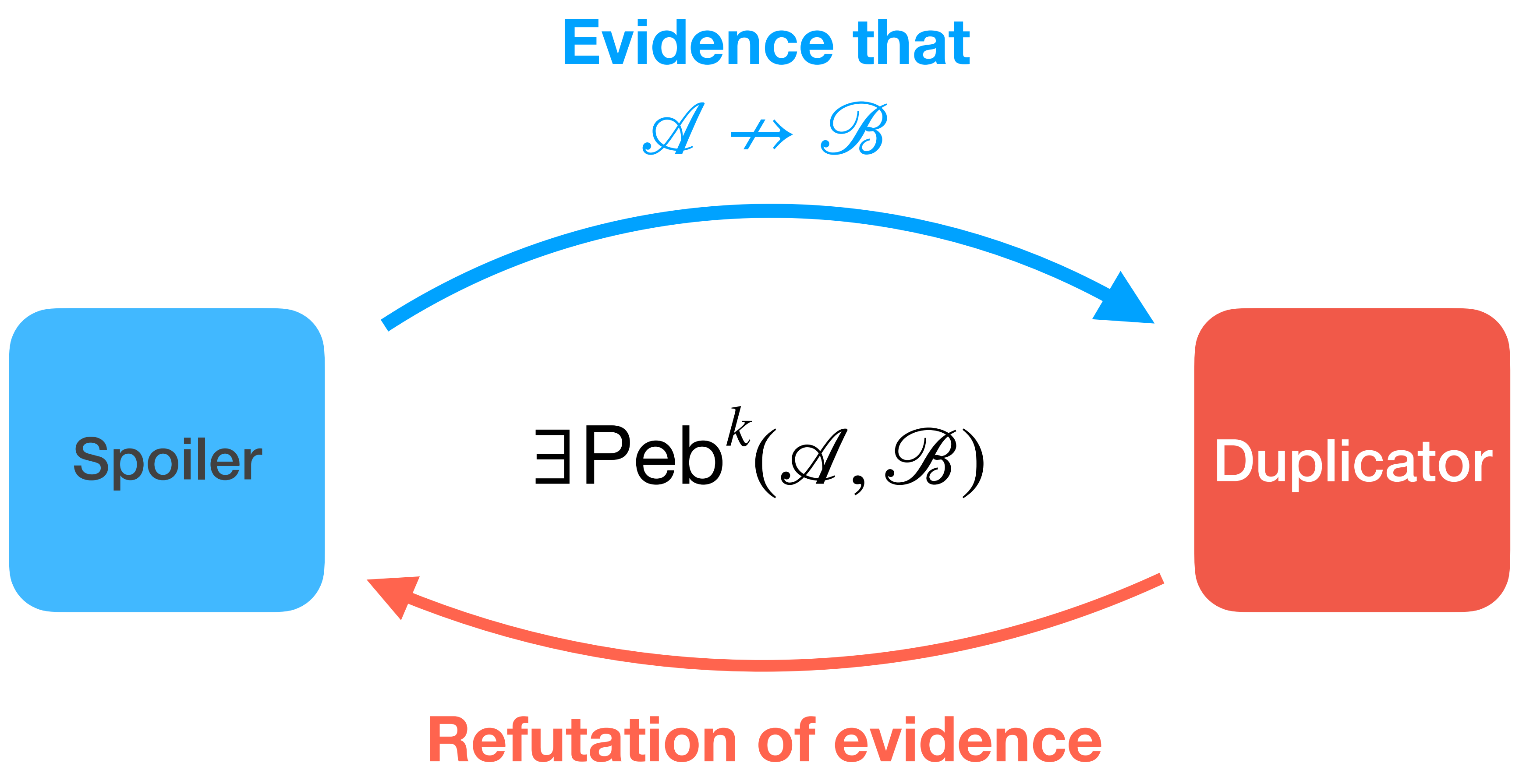
Duplicator winning implies that
$\mathscr{A} \equiv_{\mathscr{L}_k} \mathscr{B}$

Harder game for Duplicator means more expressive $\mathscr{L}_k$

One-way variants also important

- Expressiveness upper bounds :
  $\{\mathscr{A}_k\}$ all with P, $\{\mathscr{B}_k\}$ all lacking P
  Show that $\mathscr{A}_k \equiv_{\mathscr{L}_k} \mathscr{B}_k$ then P
  inexpressible in $\mathscr{L} = \cup \mathscr{L}_k$

- Success of algorithms:
  For one-way k-pebble game,
  Duplicator wins iff k-local consistency algorithm says CSP
  $\mathscr{A} \to \mathscr{B}$ has solution



Duplicator winning implies that
$\mathscr{A} \Rrightarrow_{\mathscr{L}'_k} \mathscr{B}$

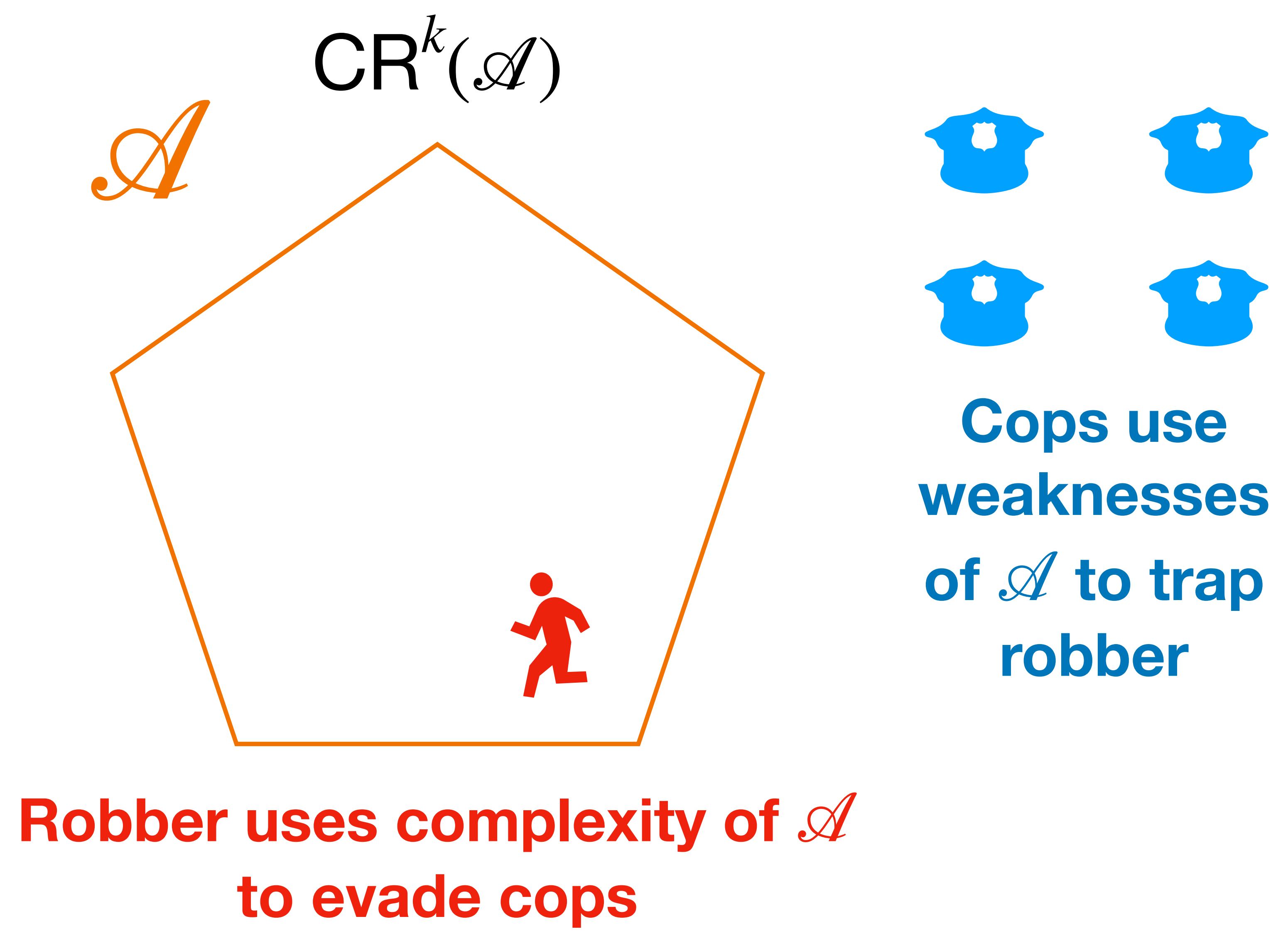
Harder game for Duplicator
means more expressive $\mathscr{L}'_k$

… and many other types of games exist!

- <u>Expressiveness upper bounds :</u>
  $\{\mathscr{A}_k\}$ all with P, $\{\mathscr{B}_k\}$ all lacking P
  Show that $\mathscr{A}_k \equiv_{\mathscr{L}_k} \mathscr{B}_k$ then P inexpressible in $\mathscr{L} = \cup \mathscr{L}_k$

- <u>Success of algorithms:</u>
  For one-way k-pebble game, Duplicator wins iff k-local consistency algorithm says CSP $\mathscr{A} \to \mathscr{B}$ has solution

- <u>Proving a structure decomposes:</u>
  Game played on one structure between a “robber” and k “cops” is won by cops when $\mathscr{A}$ has treewidth $< k$



Cops winning implies that $\mathscr{A}$ has a decomposition

Harder game for cops means simpler decomposition of $\mathscr{A}$

Game comonads: the story so far

History of game comonads

- Abramsky, Dawar & Wang, 2017
  $\mathbb{P}_k\mathscr{A}$ construction which put a relational structure on the tree of histories of Spoiler moves in $\exists\mathbf{Peb}^k(\mathscr{A}, -)$
- This turned out to be a comonad!
- Its Kleisli category relates $\exists\mathbf{Peb}^k(\mathscr{A}, \mathscr{B})$ and $\mathbf{Bij}^k(\mathscr{A}, \mathscr{B})$
- Its coalgebras correspond to winning strategies for cops in $\mathbf{CR}^k(\mathscr{A})$

$\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$\iff$

Duplicator wins $\exists$**Peb**$^k(\mathscr{A}, \mathscr{B})$

$$\mathbb{P}_k$$

$\mathscr{A} \cong_{\mathscr{K}(\mathbb{P}_k)} \mathscr{B}$

$\iff$

Duplicator wins Bij$^k(\mathscr{A}, \mathscr{B})$

$\exists\ \alpha : \mathscr{A} \to \mathbb{P}_k\mathscr{A}$ **a coalgebra**

$\iff$

Cops win CR$^k(\mathscr{A})$

Developments in game comonads

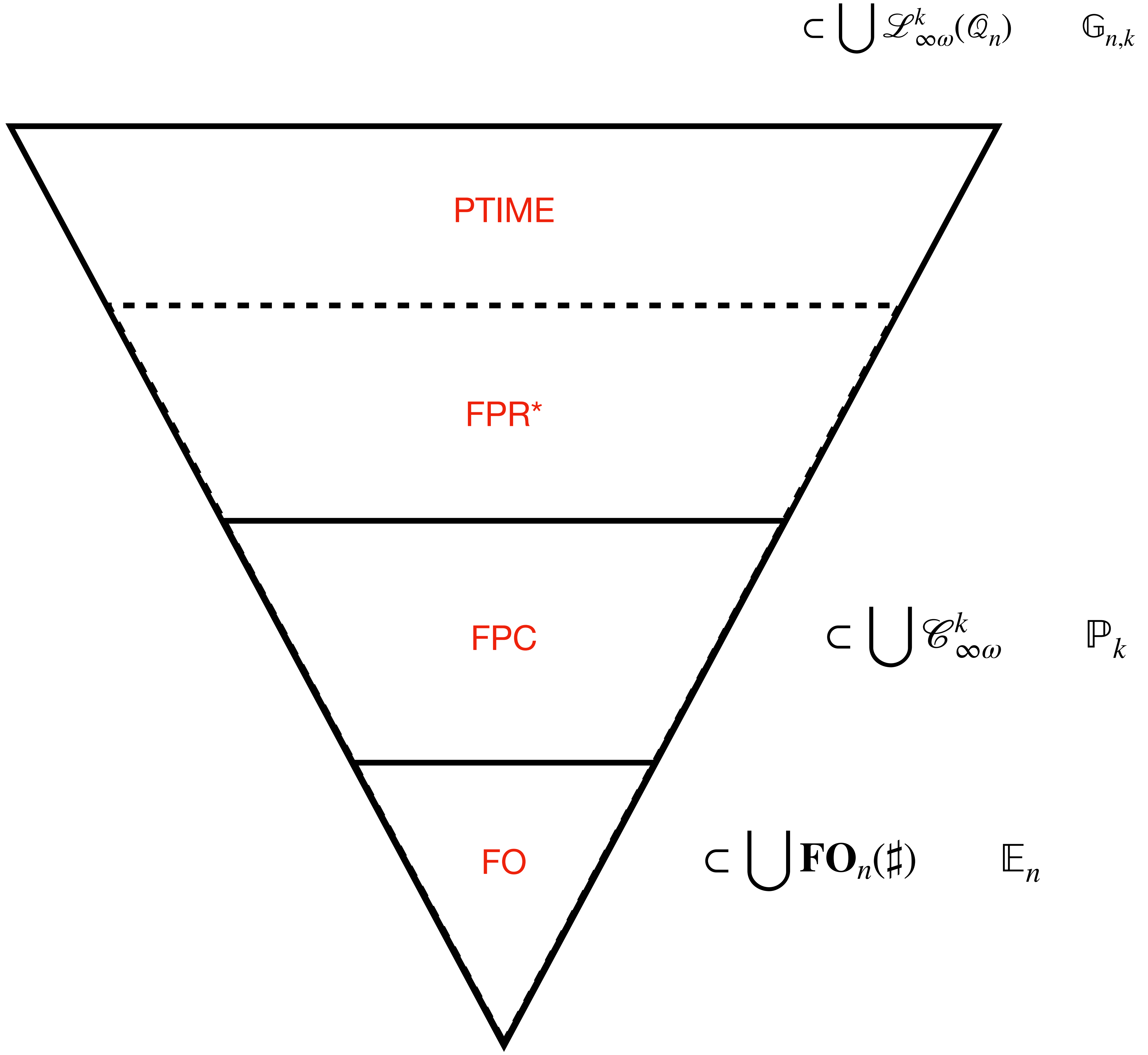
Reference	Comonad	Related games	Logical Resource	Structural constant
Abramsky, Dawar & Wang, 2017	$\mathbb{P}_k$	Pebble games	Variables	Treewidth
Abramsky & Shah, 2018	$\mathbb{E}_n$	Ehrenfeucht-Fraïssé	Quantifier depth	Treedepth
Abramsky & Shah, 2018	$\mathbb{M}_d$	Modal bisimulation	Modal depth	Modal unfolding depth
Ó Conghaile & Dawar, 2021	$\mathbb{G}_{n,k}$	Generalised quantifier games	Lindstrom quantifiers of fixed arity	Extended tree depth

And others have been created for guarded logics and pathwidth.

Rank logic and its games

Linear algebra in the search for PTIME

- In terms of logics we can work with:
    - k-variable fixed point logic with counting doesn't capture P (CFI construction)
    - For any k fixed point logic extended with all n-ary Lindstrom quantifiers doesn't capture P (Hella 1993)
- But we know that if there is a logic for P it is FPC extended with *some* vectorised family of Lindstrom quantifiers (Dawar, 1994)
- One of the two leading contenders for a logic for PTIME is fixed point logic extended with rank quantifiers



Fixed point with rank

- This logic takes $\mathbf{FO}$ extended with least fixed point operators and extends it with a further family of quantifiers $\mathbf{rk}^{\geq r}_{m,q}$ which binds $\mathbf{x}, \mathbf{y}$ in $\Psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ (which are each m-tuples of free variables)

- $\Psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defines a $|A|^m$ x $|A|^m$ 0-1 matrix indexed by m-tuples of A

$\mathscr{A}, \mathbf{a} \models \Psi(x_0, y_0, \mathbf{z})$

$\mathscr{A}, \mathbf{a} \models \neg\Psi(x_1, y_0, \mathbf{z})$

$\vdots$

$$M_\Psi := \begin{pmatrix} 1 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$\mathscr{A}, \mathbf{a} \models \mathbf{rk}^{\geq r}_{m,q}\Psi(\mathbf{x}, \mathbf{y}, \mathbf{z})$

$\Longleftrightarrow$

M_Ψ has rank $\geq r$ over $\mathbb{F}_q$

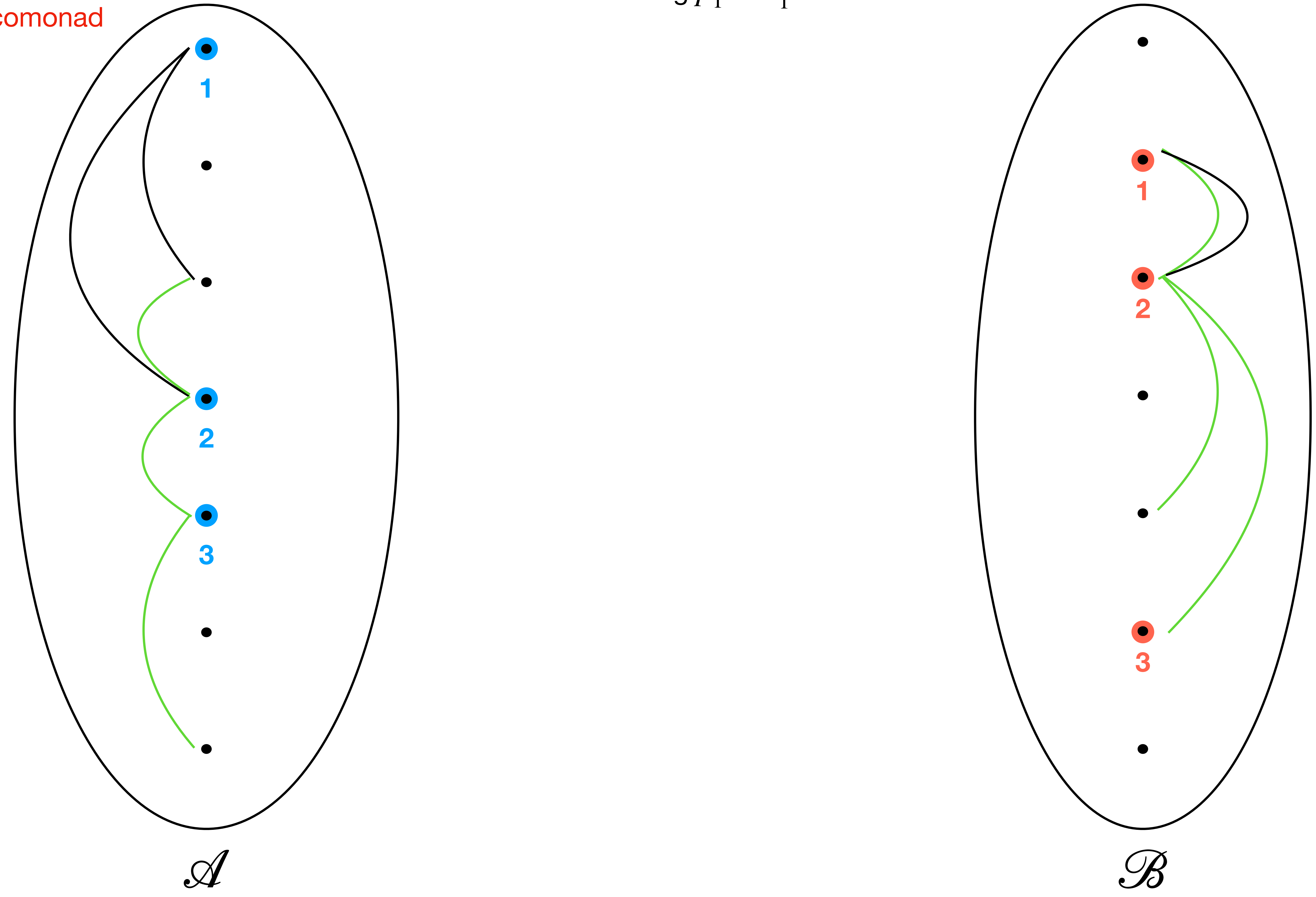
Brief history of fixed point with rank

- FPR introduced by Dawar, Grohe, Holm, Laubner 2009
- Matrix equivalence and IM-games, Dawar & Holm 2012
- *Rank logic is dead long live rank logic,* FPR*, Grädel & Pakusa 2015
- IM-games and linear algebraic logic $\mathbf{LA}^k$, Dawar, Grädel, Pakusa 2019

$\exists\mathbf{Peb}^k(\mathscr{A}, \mathscr{B})$: Duplicator responds to Spoiler "*in real time*"

- Logic: $\exists^+\mathscr{L}^k_{\infty\omega}$
- Lists of Spoiler moves have tree structure
- Game captured by $\mathbb{P}_k$ comonad

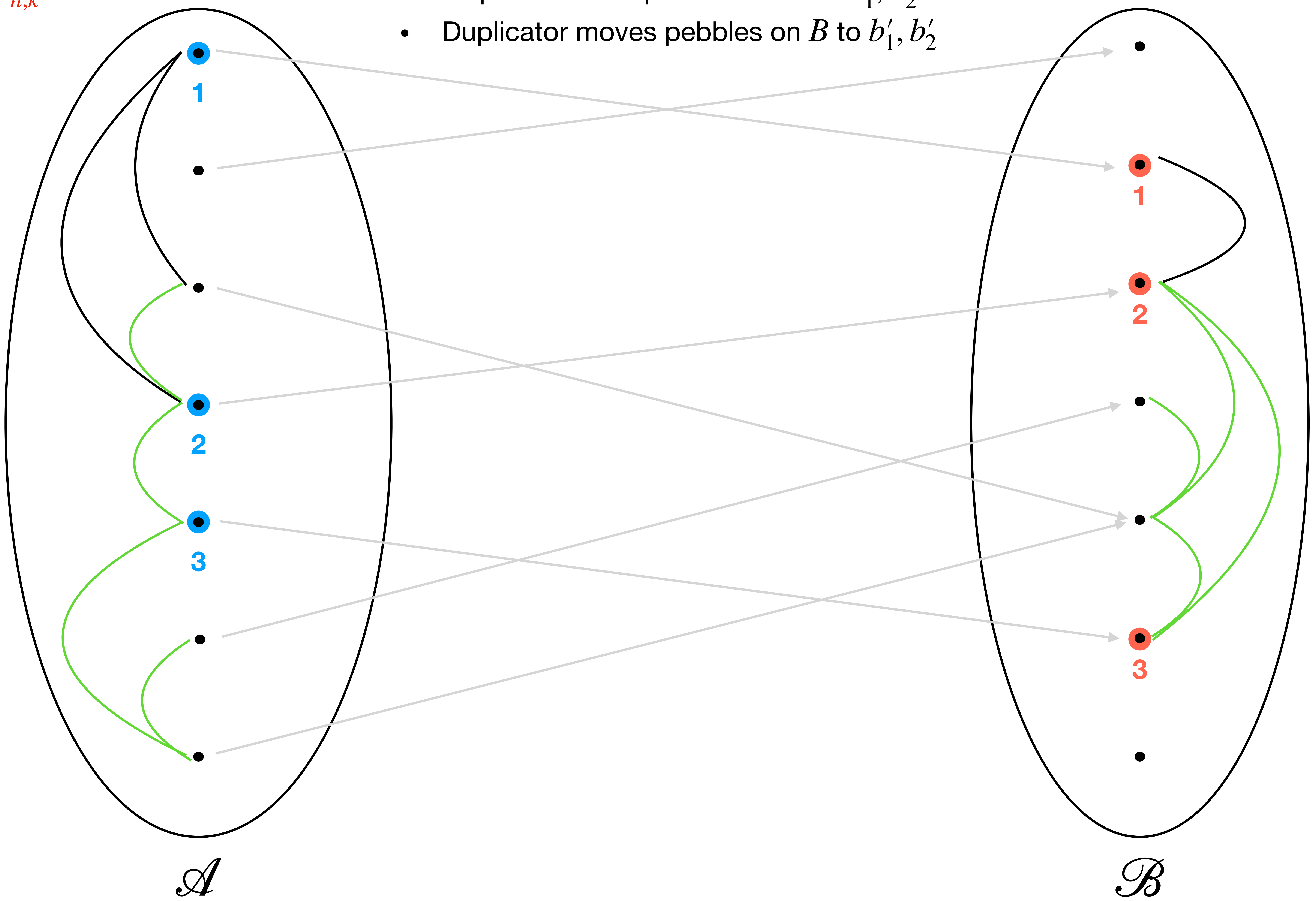
- Spoiler moves p_1 to a'_1
- Duplicator responds by moving p_1 to b'_1



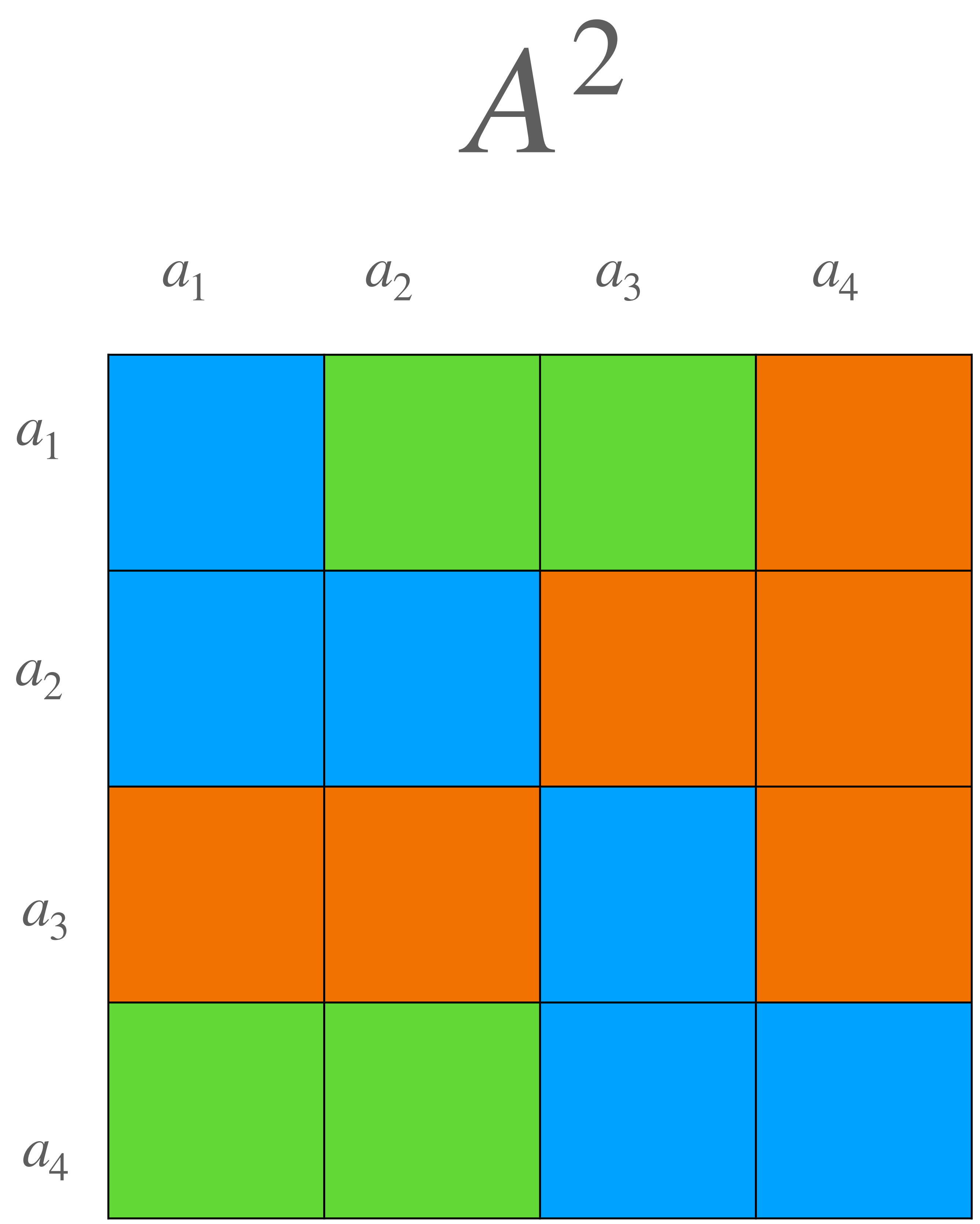
$\exists\mathbf{Fun}_n^k(\mathscr{A}, \mathscr{B})$: Duplicator responds to Spoiler "*in advance*"

- Logic: $\mathscr{L}^k_{\infty\omega}(\mathcal{Q}^h_n)$
- Lists of Spoiler moves have quotiented tree structure
- Game captured by $\mathbb{G}_{n,k}$ comonad

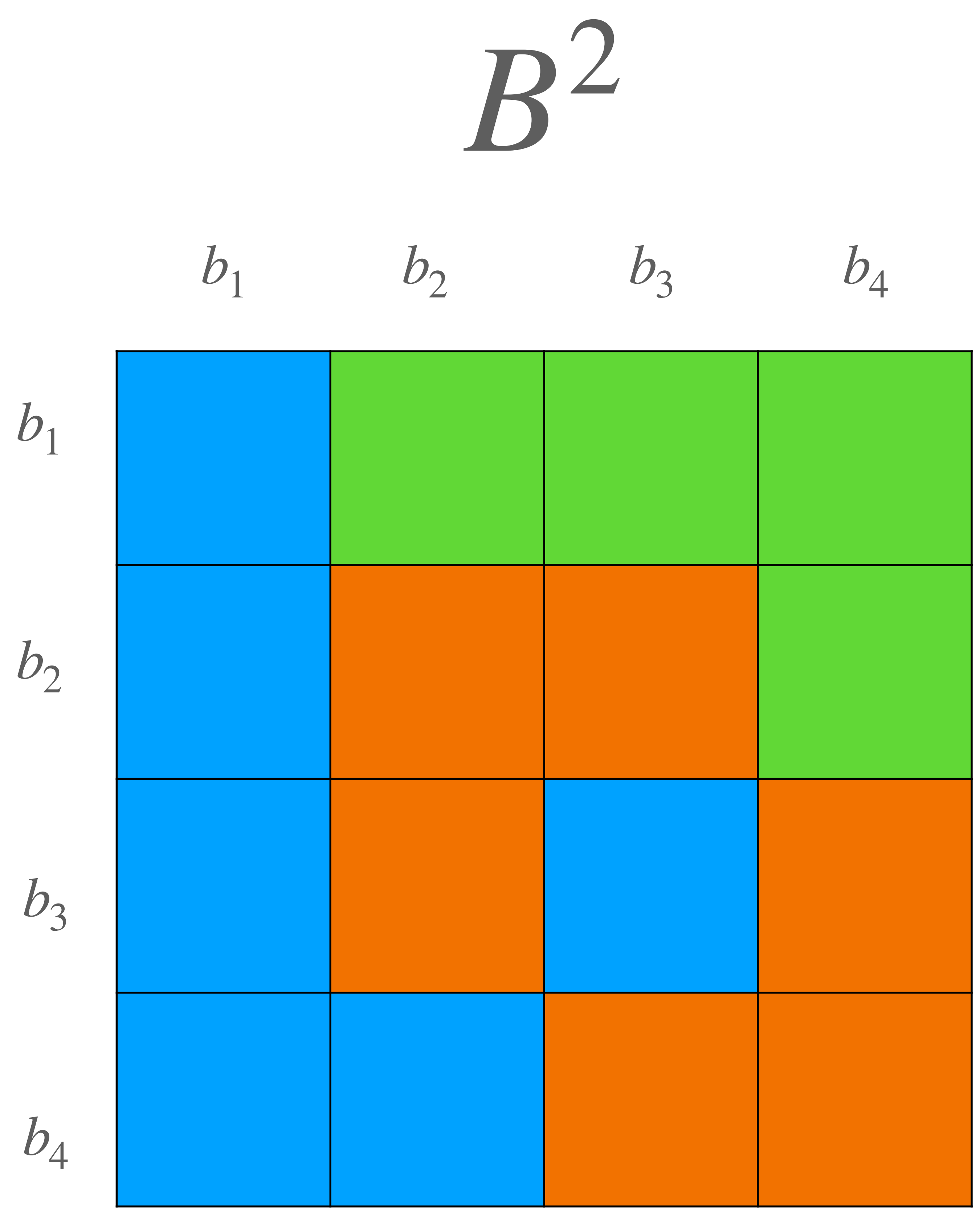
- Spoiler announces he will move p_1 and p_2
- Duplicator responds with a function $f : A \to B$
- Spoiler moves pebbles on A to a'_1, a'_2
- Duplicator moves pebbles on B to b'_1, b'_2



Partition games: in between these two

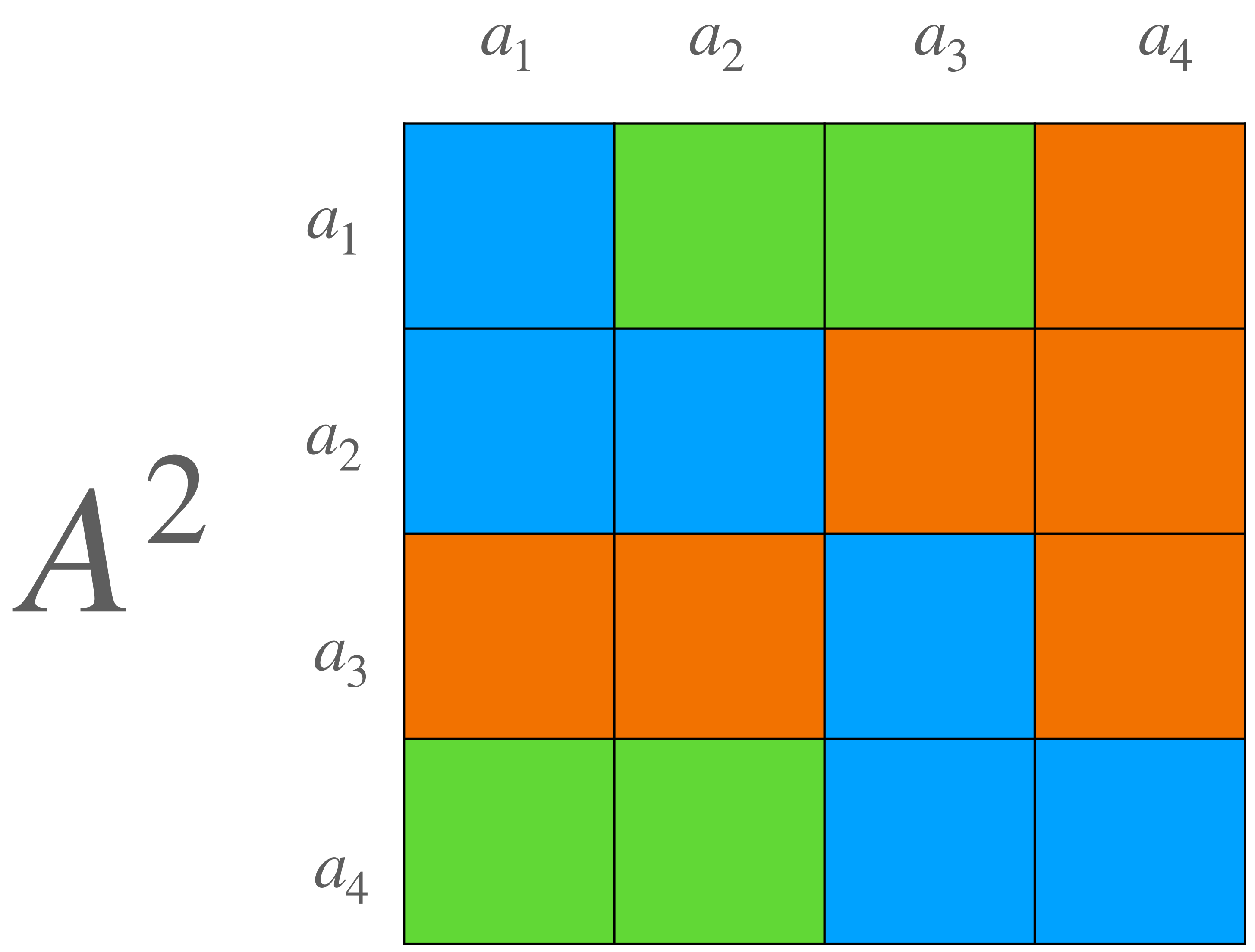


- Spoiler announces the list of m pebbles $\langle p_1, p_2, \ldots p_m \rangle$ he intends to move
- Duplicator provides partitions P, Q of A^m, B^m and bijection f between the parts of each
- Spoiler moves pebbles on A to $\langle \alpha_1, \alpha_2, \ldots \alpha_n \rangle$ **and** moves corresponding pebbles on B to some list in $f([\langle \alpha_1, \alpha_2, \ldots \alpha_n \rangle])$

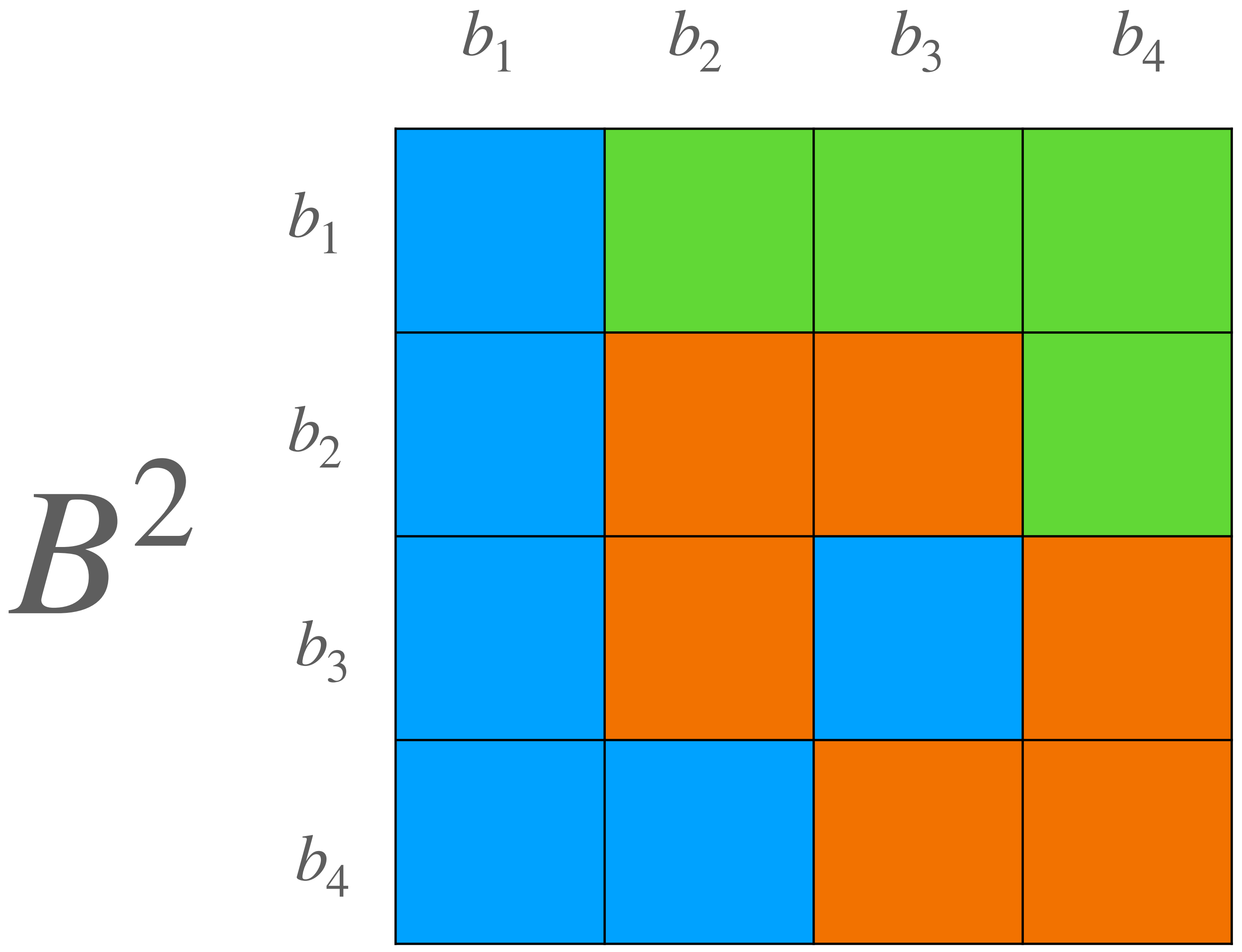


With no constraints on the partitions available to Duplicator this makes the game easier than the normal pebble game

Restricting partitions using linear algebraic constraints



$$M_1^P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad M_2^P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad M_3^P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$



$$M_1^Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad M_2^Q = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_3^Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Linear algebraic conditions on $(M_1, \ldots M_n)$

- **Matrix equivalence condition:**
  For any prime q and $\gamma : [n] \to \mathbb{F}_q$

$$\mathbf{rank}(\gamma_1 M_1^P + \ldots + \gamma_n M_n^P) = \mathbf{rank}(\gamma_1 M_1^Q + \ldots + \gamma_n M_n^Q) \quad \text{in } \mathbb{F}_q$$

Duplicator wins $\mathrm{ME}_n^k(\mathscr{A}, \mathscr{B})$

$\Longleftrightarrow$

$\mathscr{A} \equiv_{FPR_n^k} \mathscr{B}$

(Dawar & Holm)

- **Invertible-map condition:**
  There is an $A^m \times B^m$ invertible matrix S over $\mathbb{F}_q$ s.t. for each i

$$S^{-1} M_i^P S = M_i^Q \quad \text{in } \mathbb{F}_q$$

Duplicator wins $\mathrm{IM}_n^k(\mathscr{A}, \mathscr{B})$

$\Longleftrightarrow$

$\mathscr{A} \equiv_{LA_n^k} \mathscr{B}$

(Dawar, Gradel, Paduas)

Partition game comonads?

Why finding a comonad for these partition games is difficult

1. The rules are complicated!

2. The map $\langle A, R_1, \ldots R_n \rangle \mapsto (M^{R_1}, \ldots, M^{R_n})$ is **not** a functor!

3. There are no known related one-way or cops & robbers games

Progress towards finding a comonad for these games

1. Generalised quantifiers captured in $\mathbb{G}_{n,k}$

2. One-way partition games defined (but still not fully understood)

3. Duplicator winning strategies for $\mathbf{IM}_n^k$ correspond to subsets of Kleisli homsets for $\mathscr{K}(\mathbb{P}_k)$

Open questions

1. What is the "existential positive" logic for one-way linear algebra games?

2. Is there an appropriate structural parameter that extends treewidth?
   (i.e. Cops and robbers with linear algebraic rules)

3. Is it possible to show that partition games don't behave like other game comonads?
   Abramsky & Reggio *Arboreal Categories, 2021*

Thanks for listening!