Game Comonads & Generalised Quantifiers

CSL 2021

Adam Ó Conghaile & Anuj Dawar, University of Cambridge

Talk Outline

- Logic, Complexity & Games: a (very) brief history

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- Game Comonads: logical “resources” compositionally

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- Game Comonads: logical “resources” compositionally
- Generalised Quantifiers: a very powerful “resource”
- $\mathbb{G}_{n,k}$: a game comonads for generalised quantifiers

Logic, Complexity and Games

Descriptive Complexity

A quick tour

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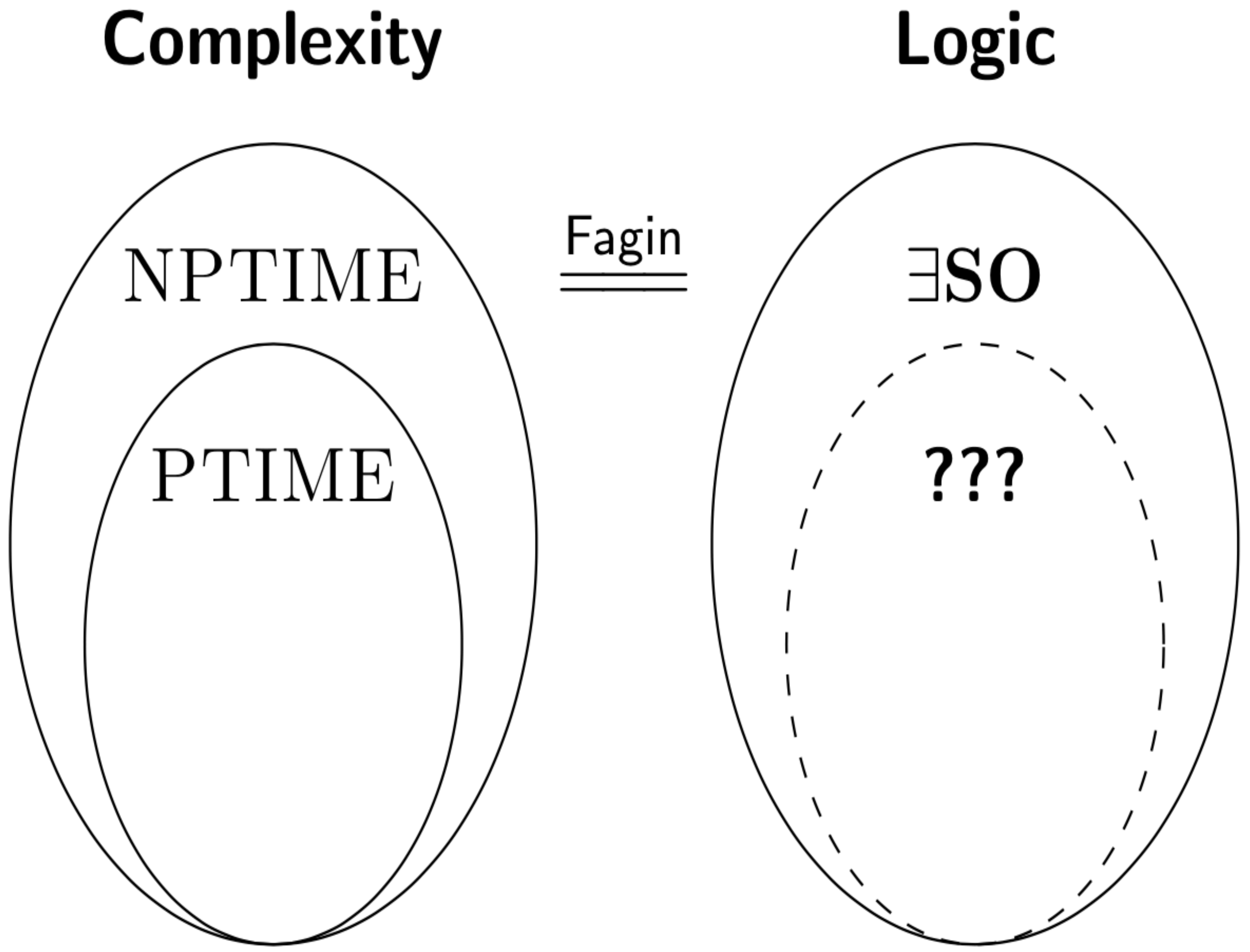
- (Fagin's Theorem, 1973)
  A class of finite structures is decidable in NP if and only if it is expressible in $\exists$SO



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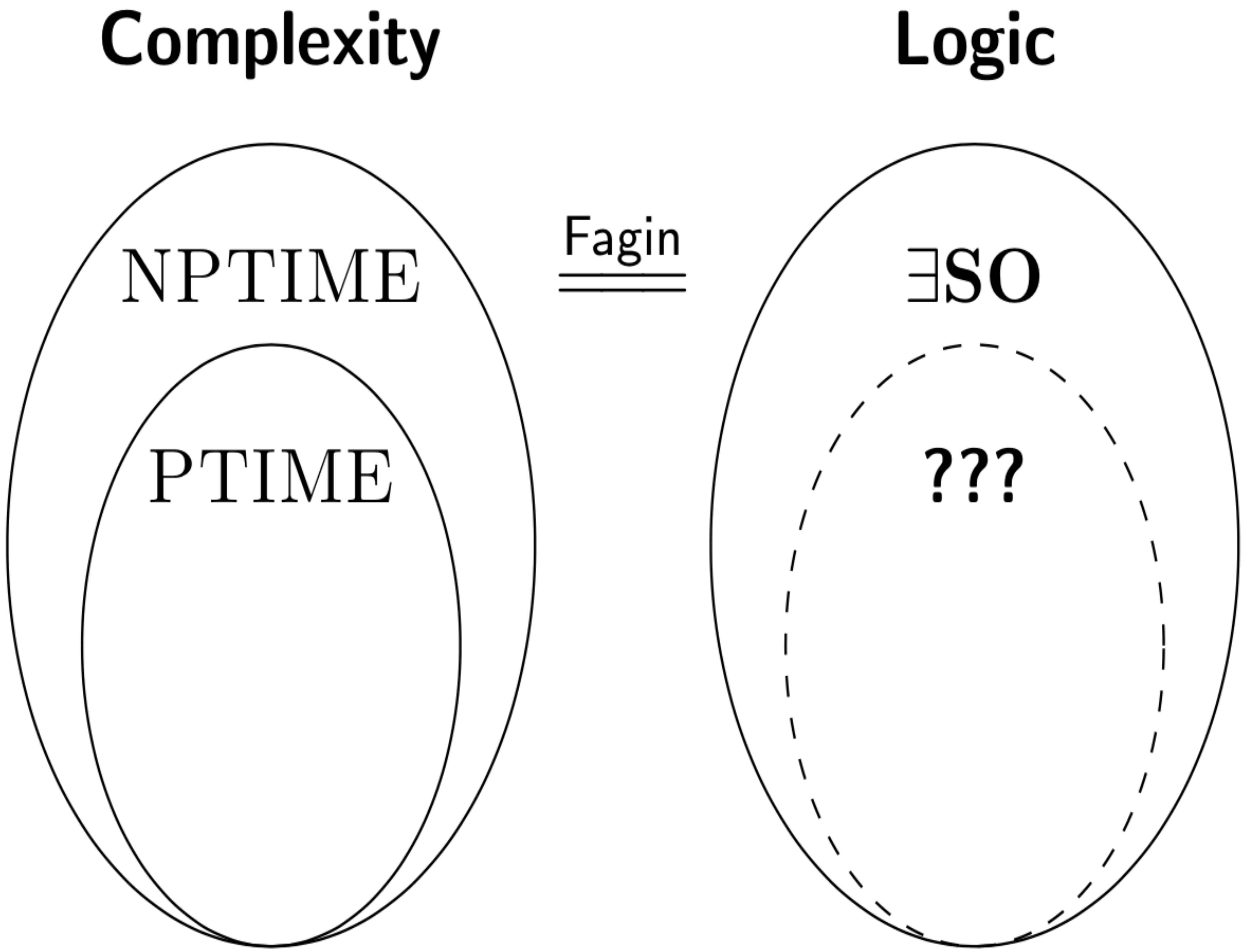
- (Gurevich's Conjecture, 1988)
  There is no equivalent logic for P



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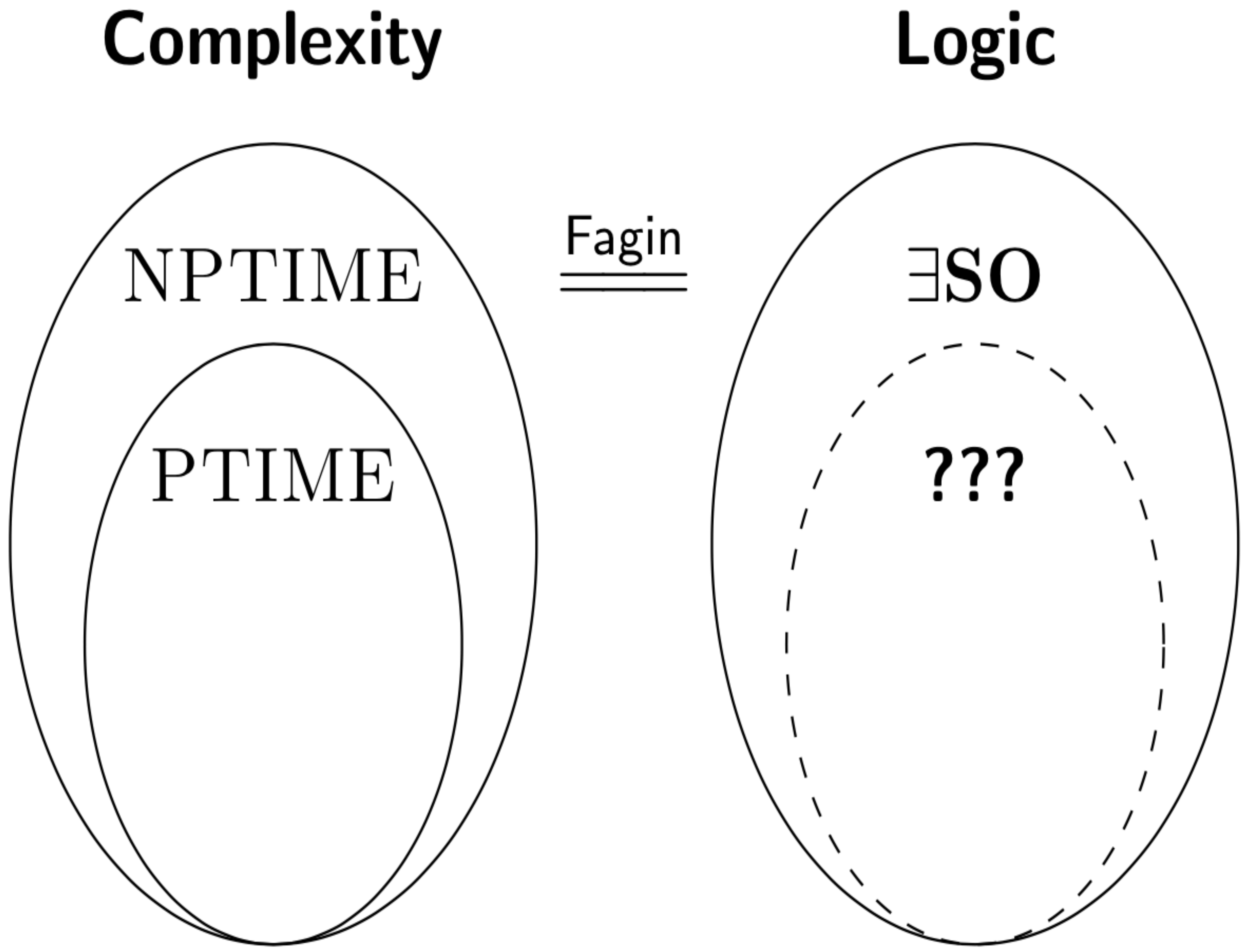
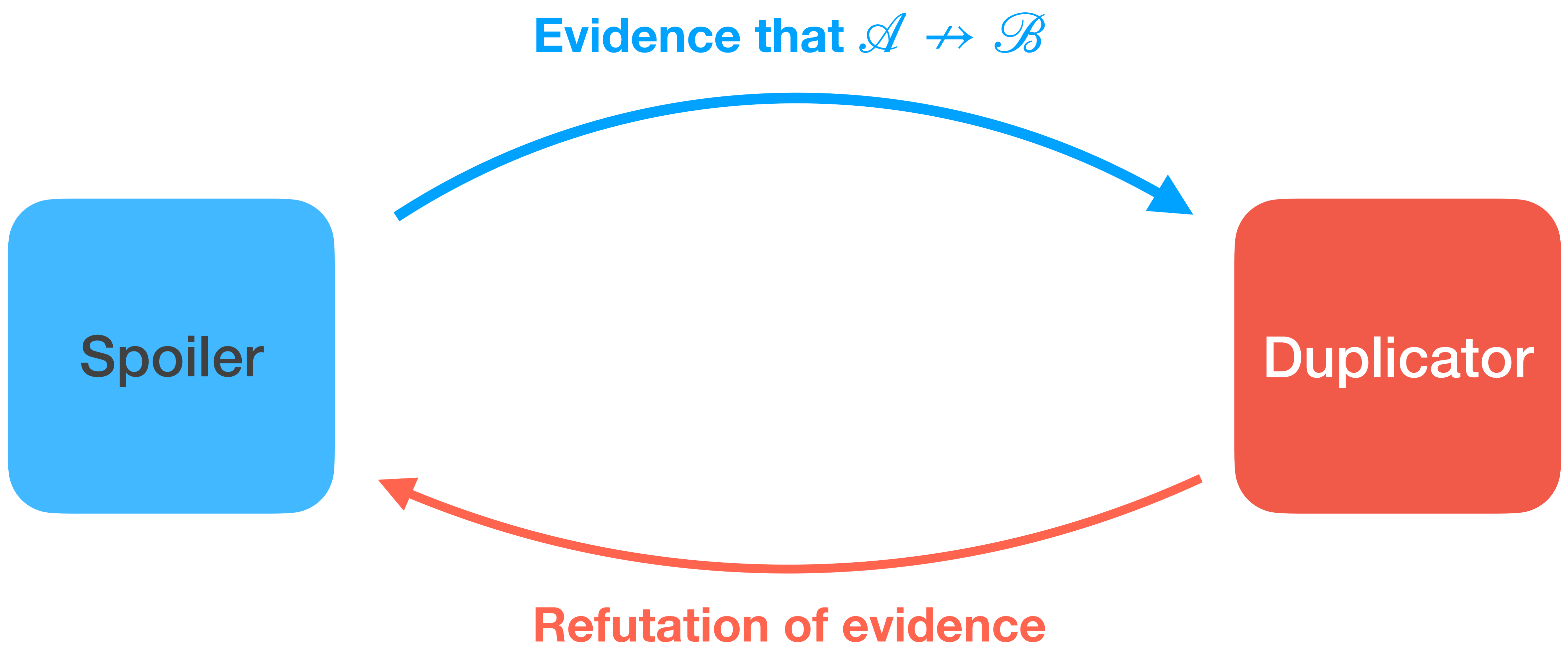
- (Fagin's Theorem, 1973)
  A class of finite structures is decidable in NP if and only if it is expressible in $\exists$SO
- (Gurevich's Conjecture, 1988)
  There is no equivalent logic for P
- Candidate logics for P include rank logic, and choiceless polynomial time.



Games: a key tool for logic

Spoiler-Duplicator Games on relational structures $\mathscr{A}, \mathscr{B}$ over signature σ



<u>One-way Games</u>

Duplicator "wins" iff $\mathscr{A} \Rrightarrow_{\mathscr{L}} \mathscr{B}$

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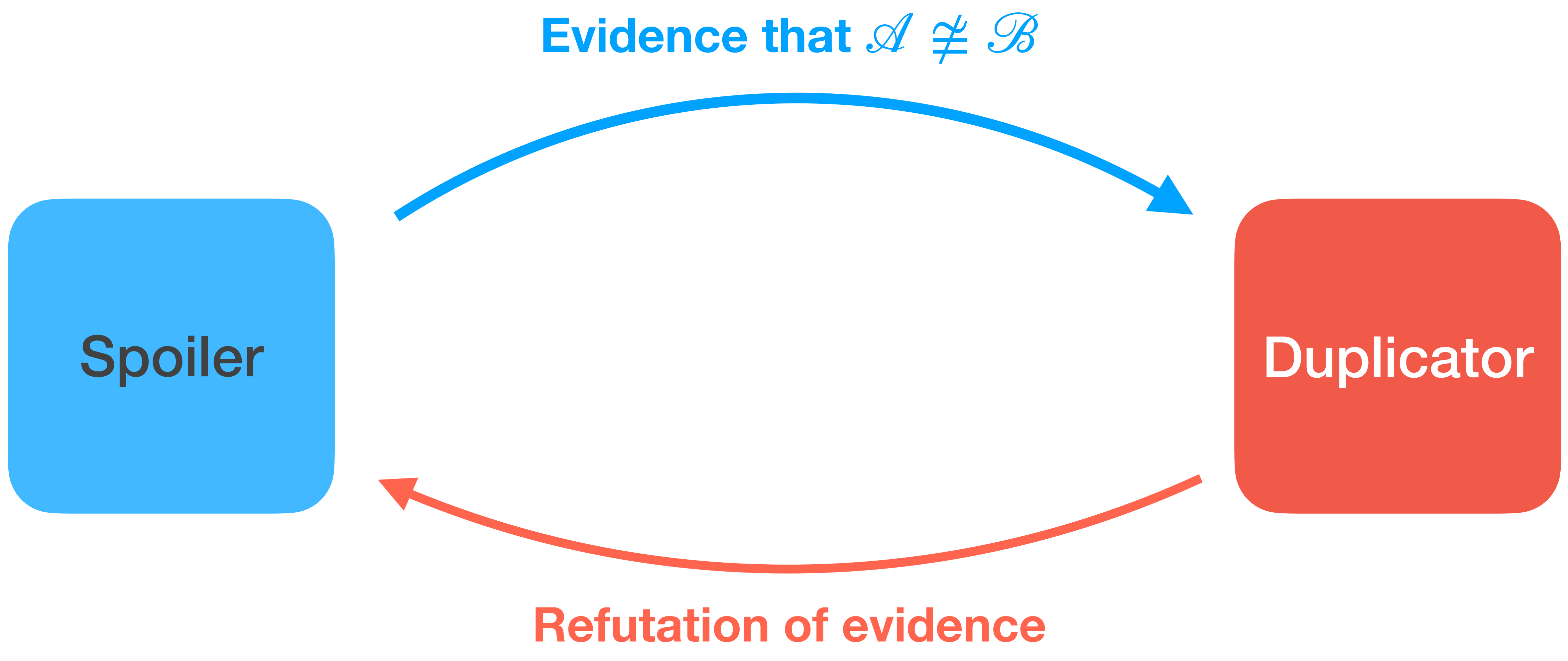


<u>One-way Games</u>

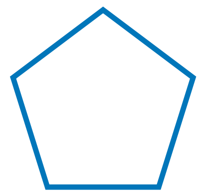
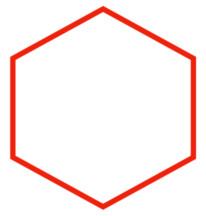
Duplicator “wins” iff $\mathscr{A} \Rrightarrow_{\mathscr{L}} \mathscr{B}$

<u>Two-way Games</u>

Duplicator “wins” iff $\mathscr{A} \equiv_{\mathscr{L}} \mathscr{B}$

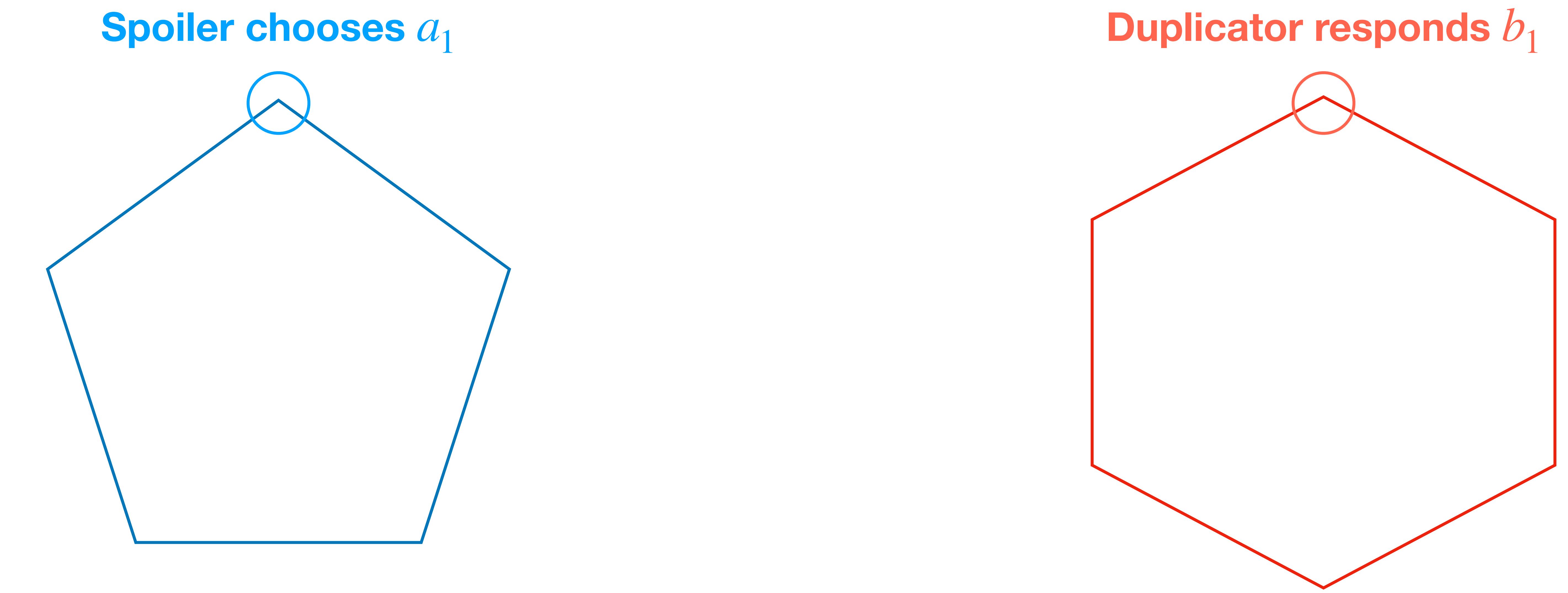
The exact $\mathscr{L}$ depends on the rules of the game

Example of Spoiler-Duplicator Games

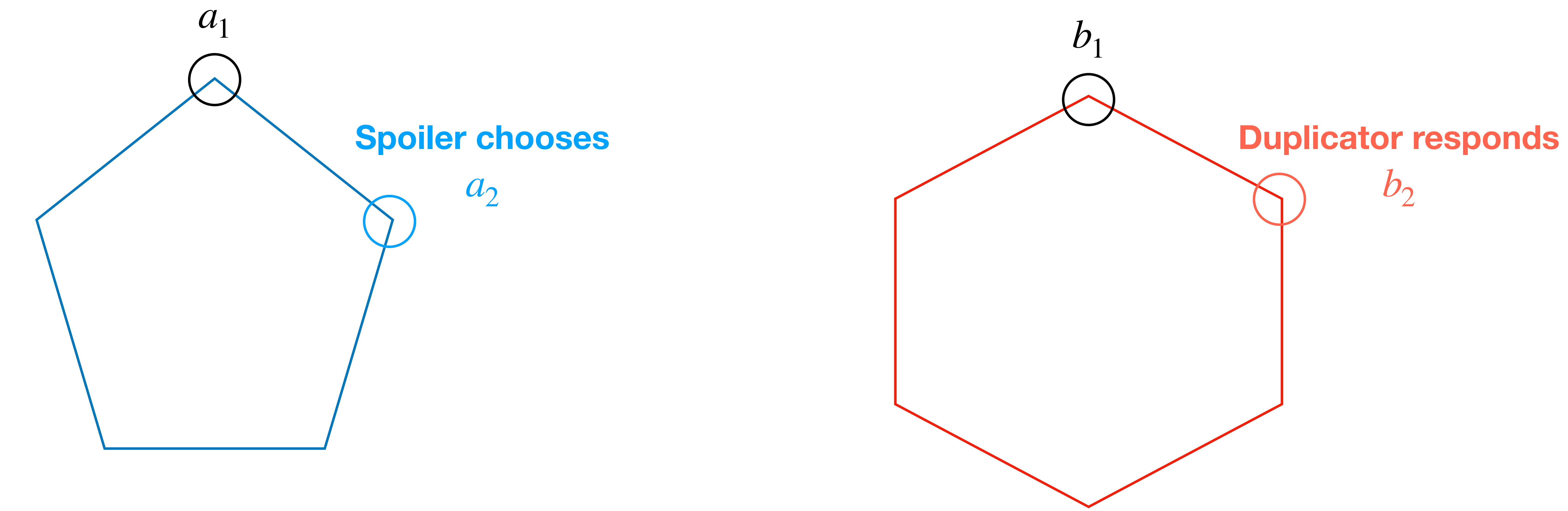
Ehrenfeucht-Fraïssé Game between  and 

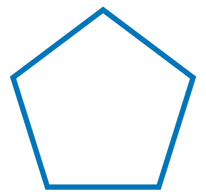
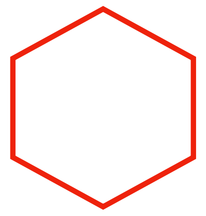
$(\sigma = \{E\})$

Round 1



Round 2



Ehrenfeucht-Fraïssé Game between  and 

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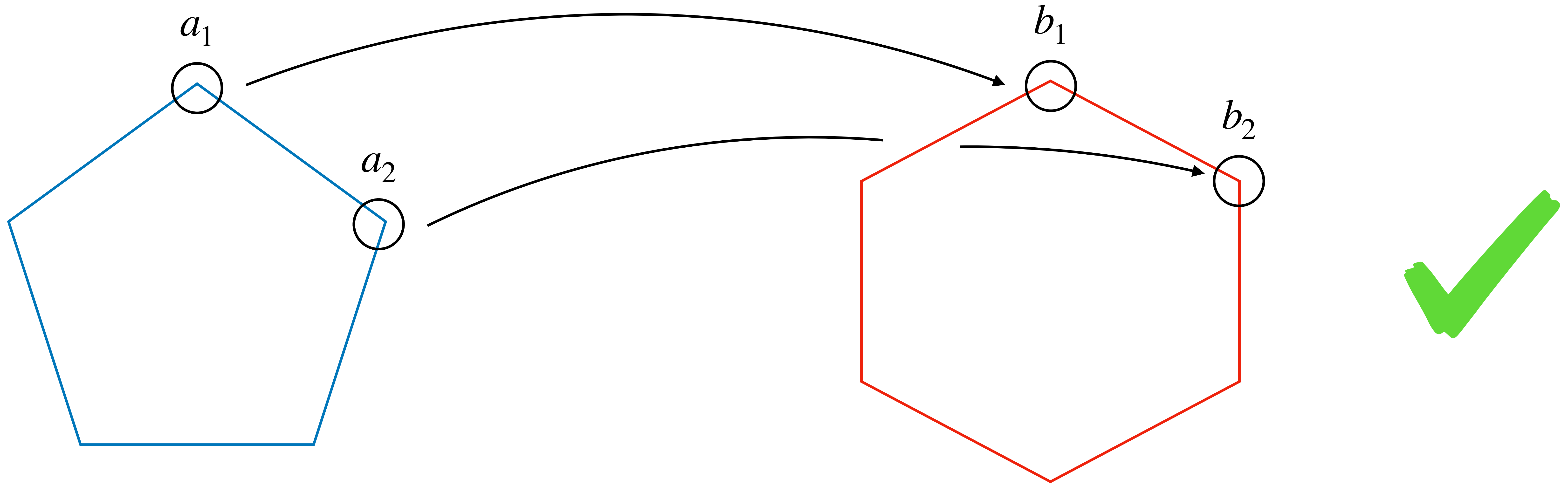
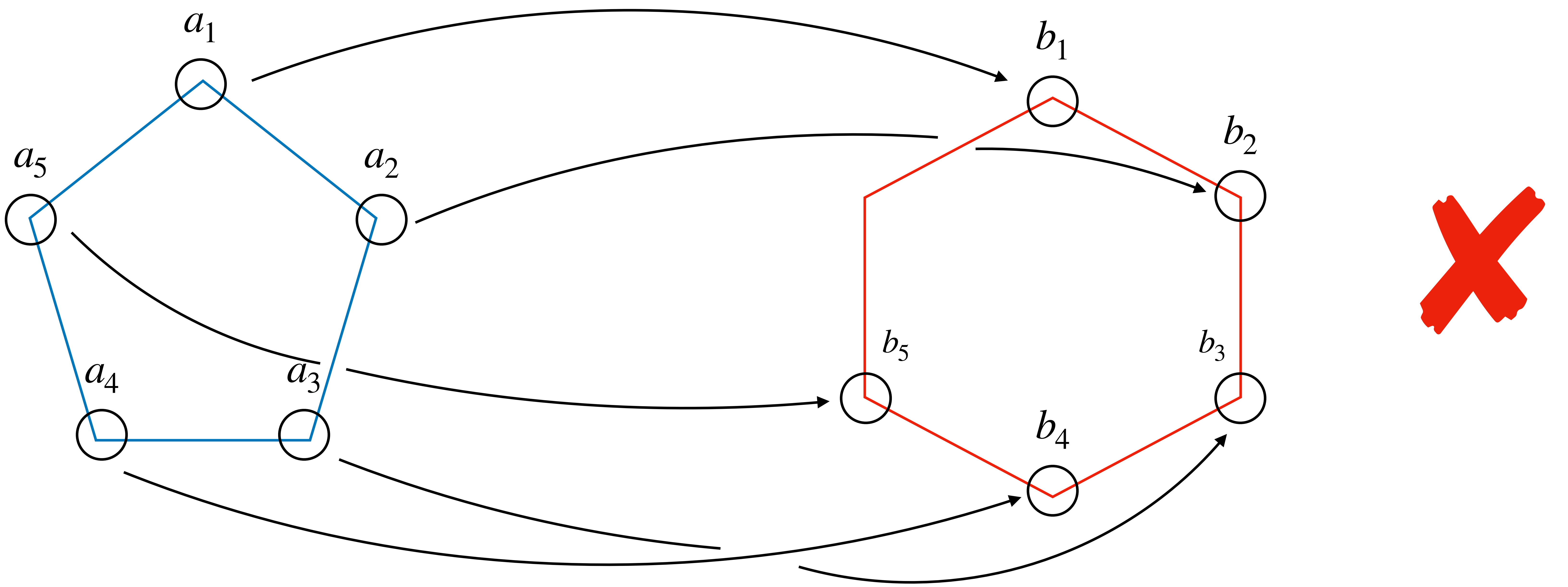


Round 5



Duplicator winning implies that $\mathscr{A}$ and $\mathscr{B}$ are related in $\mathscr{L}$

Harder game for Duplicator means more expressive $\mathscr{L}$

Reference	Game	Corresponding Logical Relation
Fraïssé 1950’s	$\exists\mathsf{EF}_k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \Rrightarrow_{\exists^+\mathscr{L}_k} \mathscr{B}$

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Hella 1996	$\mathrm{Bij}_k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \equiv_{\mathscr{C}^k} \mathscr{B}$

← $\exists^{\geq m}$

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Hella 1996	$\mathsf{Bij}_n^k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \equiv_{\mathscr{L}^k(\mathscr{Q}_n)} \mathscr{B}$ ← $Q_{\mathscr{K}}$

The Rise of Game Comonads

Can we connect these two categorically?

$$(\mathscr{R}(\sigma), \to, \cong) \rightleftarrows (\mathscr{R}(\sigma), \Rrightarrow_{\mathscr{L}}, \equiv_{\mathscr{L}})$$

Abramsky, Dawar & Wang's Pebbling Comonad

$\mathbb{P}_k\mathscr{A} = \langle (A \times [k])^+$, relations from $\mathscr{A}$ according to tree structure$\rangle$

Counit $\epsilon : \mathbb{P}_k\mathscr{A} \to \mathscr{A}$

$$\epsilon([(a_1, p_1), \ldots, (a_m, p_m)]) = a_m$$

Comultiplication $\delta : \mathbb{P}_k\mathscr{A} \to \mathbb{P}_k\mathbb{P}_k\mathscr{A}$

$$\delta([(a_1, p_1), \ldots, (a_m, p_m)]) = [(s_1, p_1), \ldots (s_m, p_m)]$$

where $s_i = [(a_1, p_1) \ldots, (a_i, p_i)]$

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Kleisli Category $\mathscr{K}(\mathbb{P}_k)$

$\mathbb{P}_k\mathscr{A} \to \mathscr{B} \iff$ Duplicator has a winning strategy for $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$

$\mathscr{A} \cong_{\mathscr{K}(\mathbb{P}_k)} \mathscr{B} \iff$ Duplicator has a winning strategy for $\mathrm{Bij}_k(\mathscr{A}, \mathscr{B})$

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Coalgebras

$\alpha : \mathscr{A} \to \mathbb{P}_k\mathscr{A} \iff \mathscr{A}$ has a tree decomposition of width k

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Where $\mathbb{P}_k$ is graded in k which controls the number of variables in the underlying logic

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Abramsky & Shah 2018	$\mathbb{M}_n$	Modal bisimulation	Modal depth

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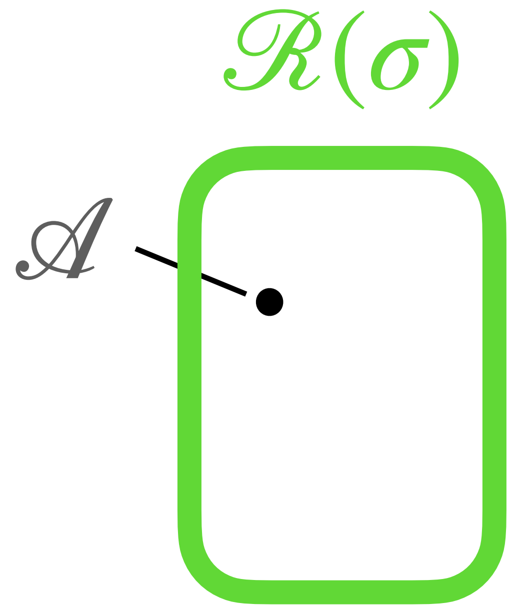
and

$\cong_{\mathscr{K}}$ is $\Rrightarrow_{\mathscr{L}(\exists^{\geq m})}$

Quantifiers as a Resource

Building a new quantifier

A relational structure $\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R \in \sigma} \rangle \in \mathscr{R}(\sigma)$



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A class of structures $\mathscr{K} \subset \mathscr{R}(\tau)$

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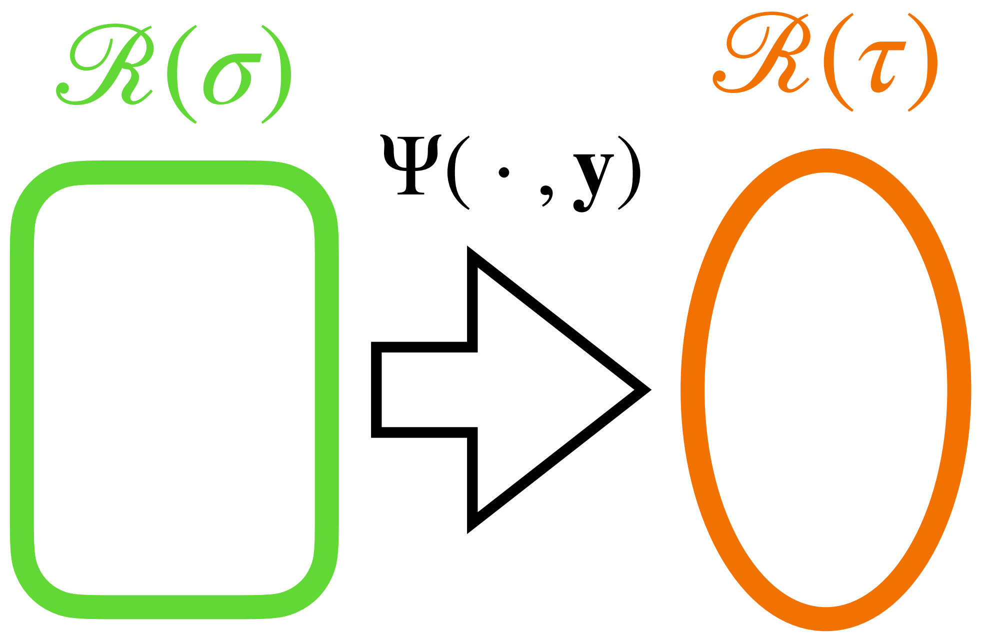
$$\mathscr{K} \subset \mathscr{R}(\tau)$$

An interpretation

$$\Psi(\mathbf{x}, \mathbf{y}) = \langle \psi_T(\mathbf{x}_T, \mathbf{y}_T)\rangle_{T\in\tau}$$



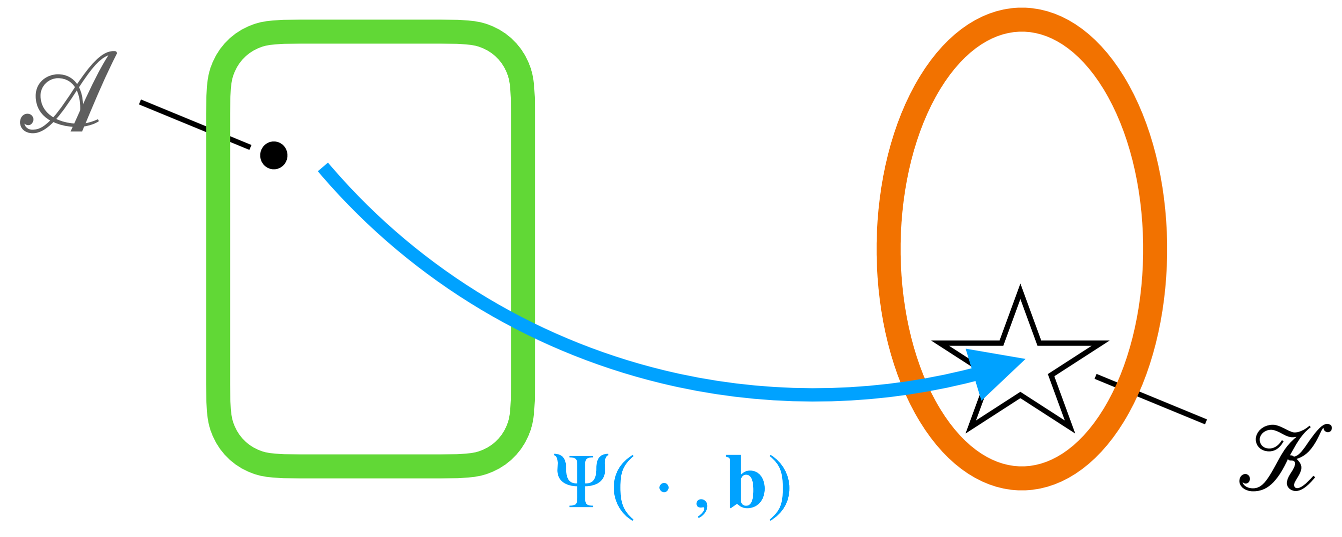
Building a new quantifier

A relational structure $\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R \in \sigma} \rangle \in \mathscr{R}(\sigma)$

A class of structures $\mathscr{K} \subset \mathscr{R}(\tau)$

An interpretation $\Psi(\mathbf{x}, \mathbf{y}) = \langle \psi_T(\mathbf{x}_T, \mathbf{y}_T) \rangle_{T \in \tau}$

A new quantifier $\mathscr{A}, \mathbf{b} \models Q_{\mathscr{K}} \mathbf{x} \,.\, \Psi(\mathbf{x}, \mathbf{y})$



Building a new quantifier

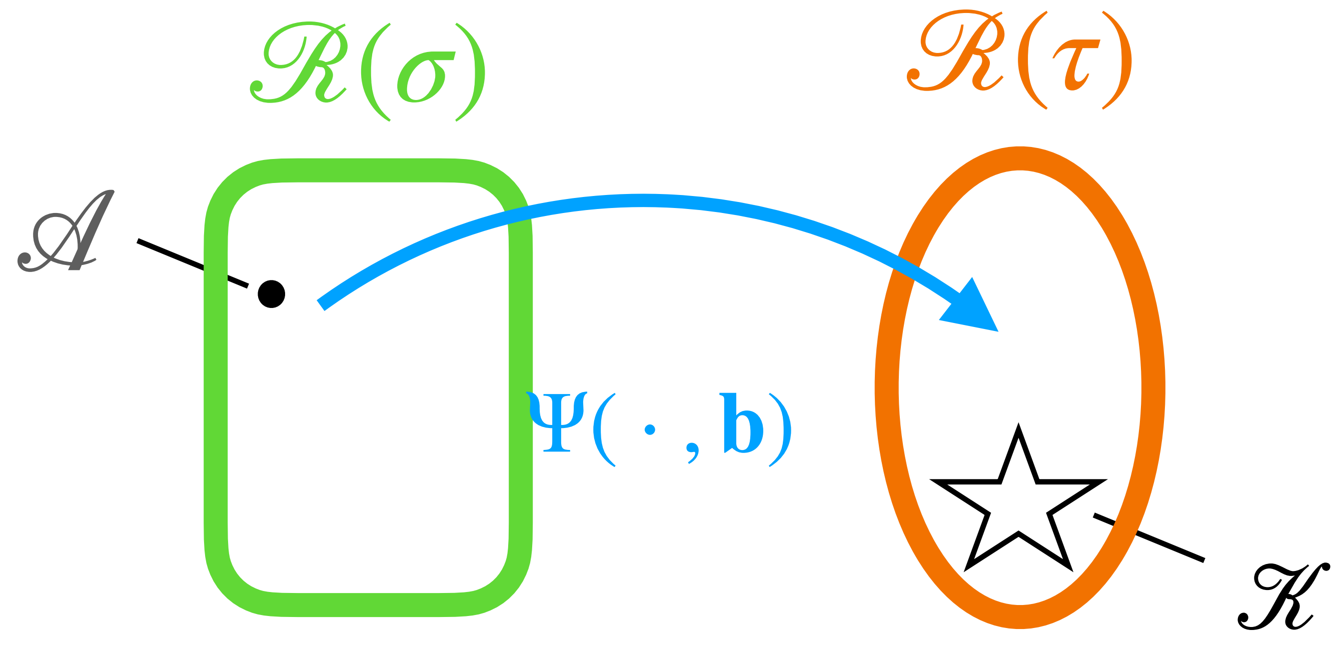
$$\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R\in\sigma}\rangle \in \mathscr{R}(\sigma)$$

$$\mathscr{K} \subset \mathscr{R}(\tau)$$

$$\Psi(\mathbf{x},\mathbf{y}) = \langle \psi_T(\mathbf{x}_T,\mathbf{y}_T)\rangle_{T\in\tau}$$

A new quantifier

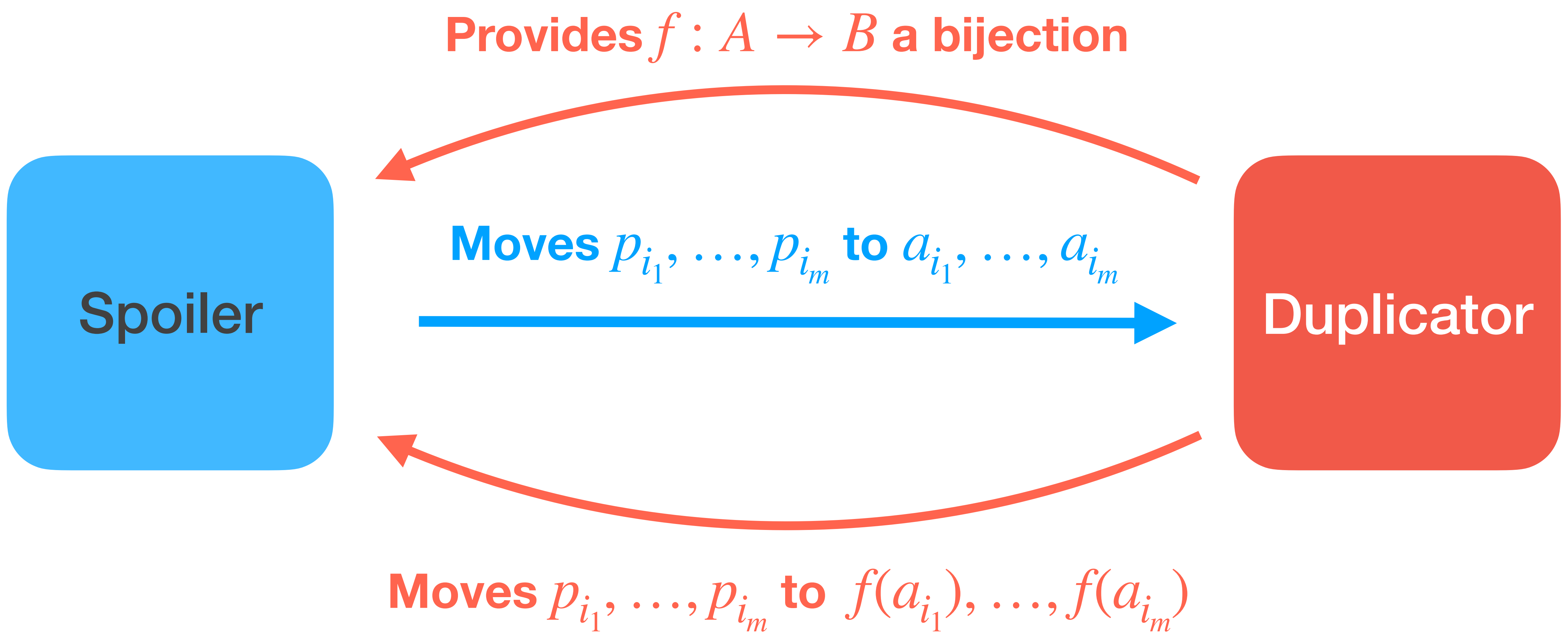
$$\mathscr{A}, \mathbf{b} \not\models Q_{\mathscr{K}}\mathbf{x} \,.\, \Psi(\mathbf{x},\mathbf{y})$$



A game to control these new quantifiers

$\mathscr{L}^k(\mathbf{Q}_n)$ is k-variable infinitary first-order logic extended by quantifiers of isomorphism-closed classes of structures with no relation of arity $> n$

$\text{Bij}_n^k(\mathscr{A}, \mathscr{B})$ game (Hella 1996)

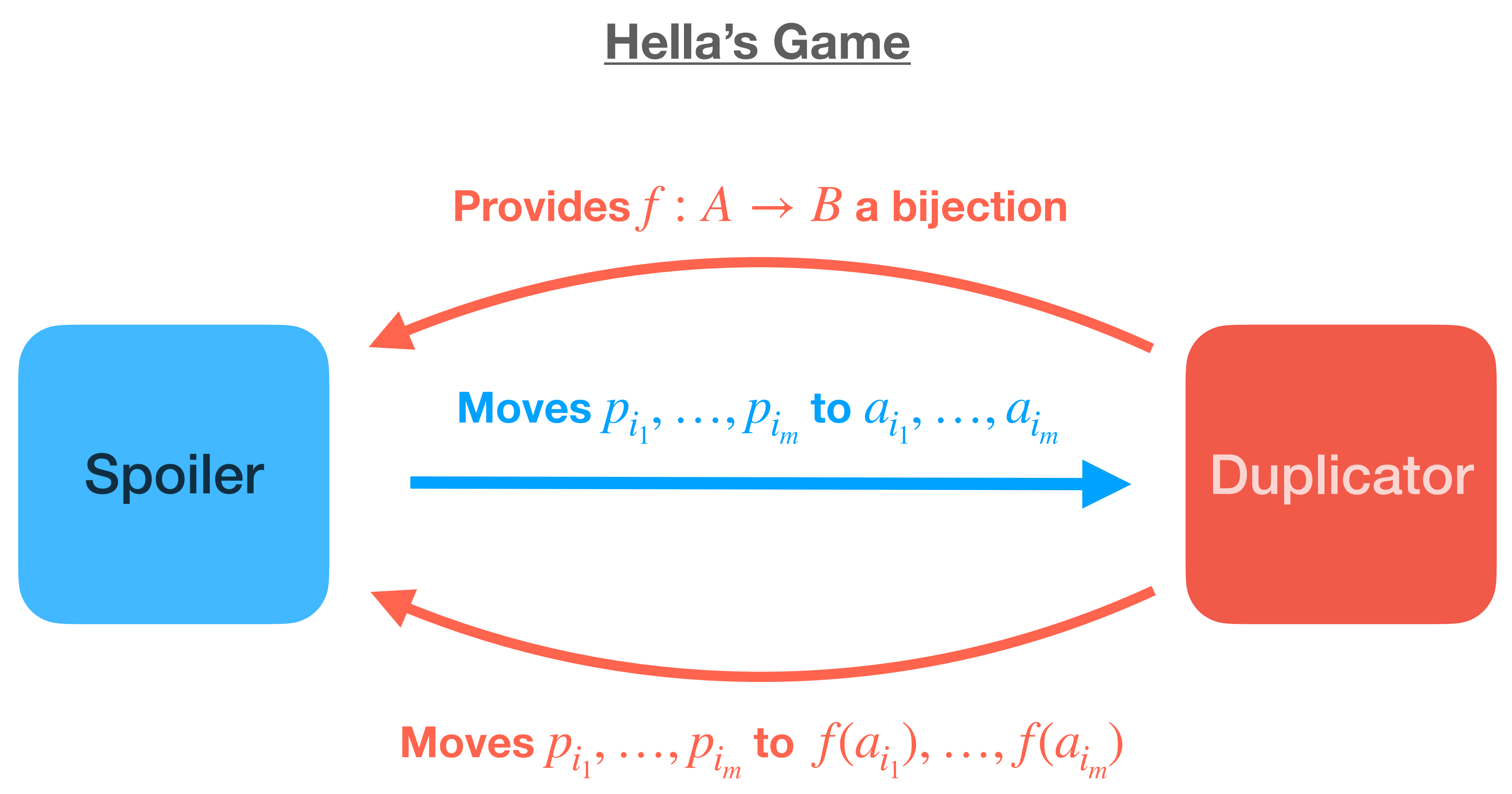


Theorem (Hella 1996)

Duplicator has a winning strategy for $\text{Bij}_n^k(\mathscr{A}, \mathscr{B})$ if and only if $\mathscr{A} \equiv_{\mathscr{L}^k(\mathbf{Q}_n)} \mathscr{B}$

$\mathbb{G}_{n,k}$: a comonad for quantifiers

Inventing new games and relating them to new logics



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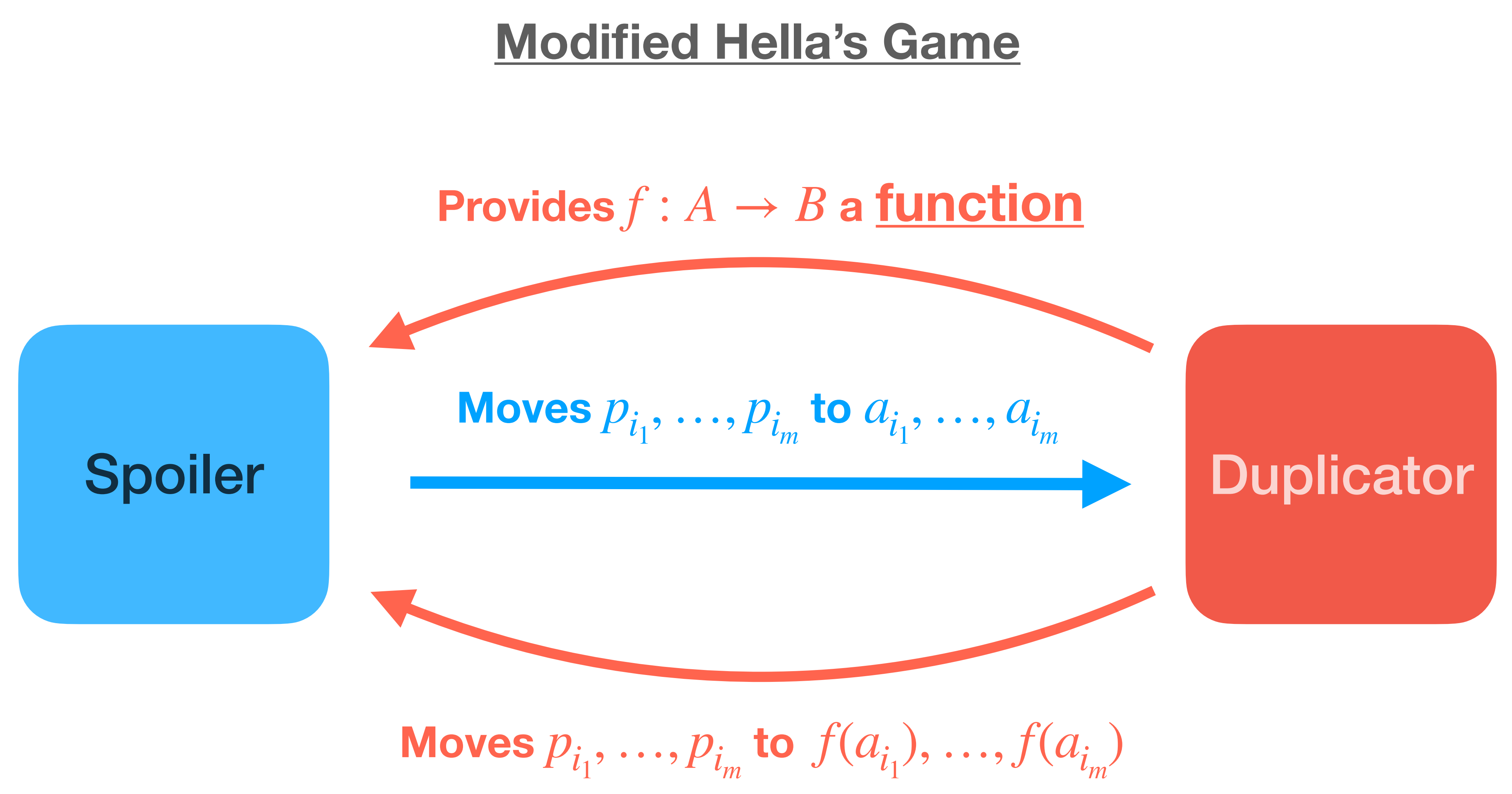
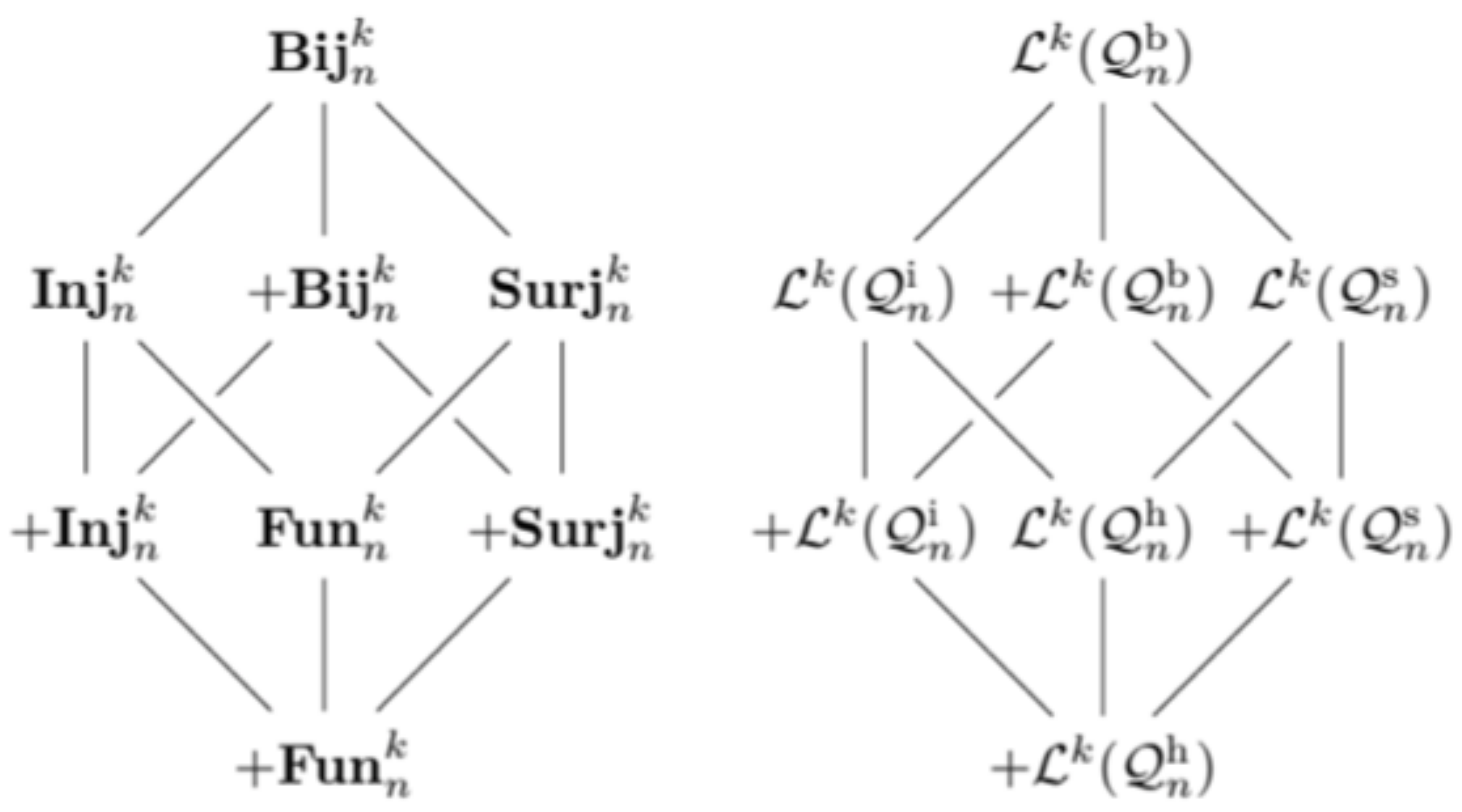


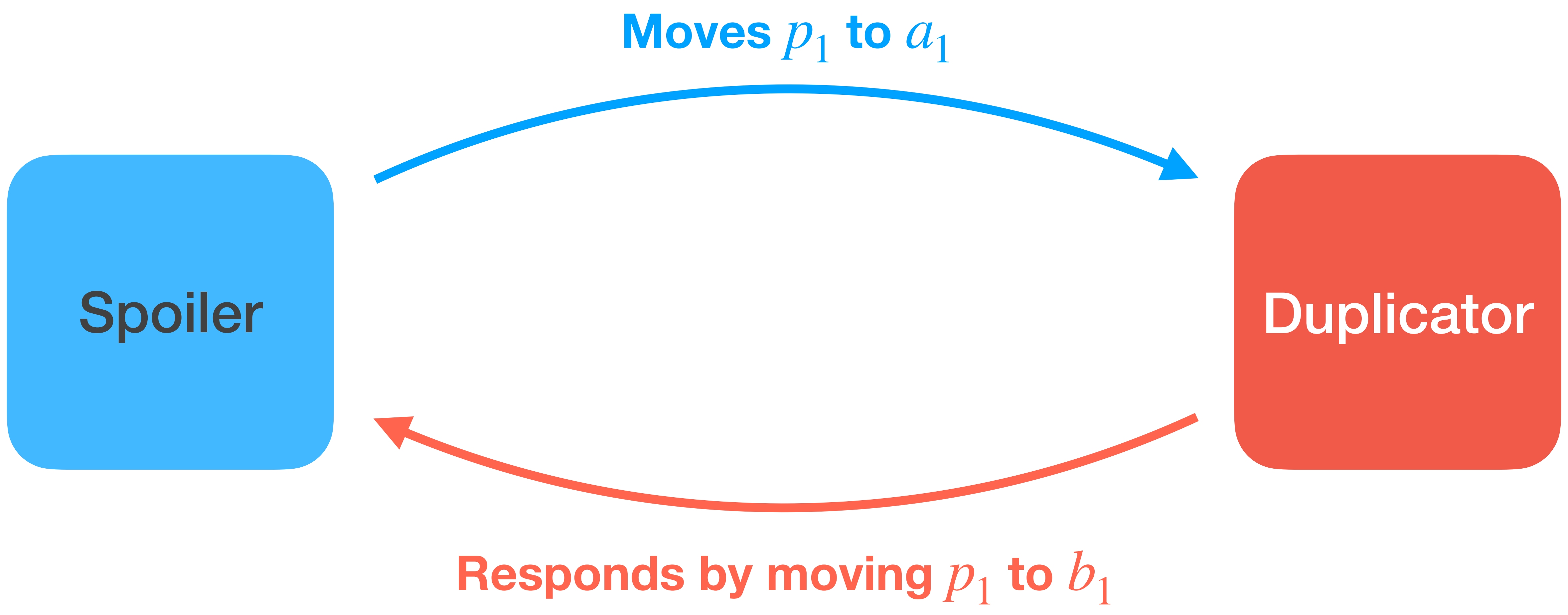


Theorem 15 (Ó C. & Dawar, 2021)

For a game $\mathscr{G}$ from the left-hand diagram, Duplicator wins $\mathscr{G}(\mathscr{A}, \mathscr{B})$ if and only if $\mathscr{A} \Rrightarrow_{\mathscr{L}^{\mathscr{G}}} \mathscr{B}$ where $\mathscr{L}^{\mathscr{G}}$ is the corresponding logic from the right-hand diagram

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$



A homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$$[(p_1, a_1)] \quad \mapsto \quad b_1$$

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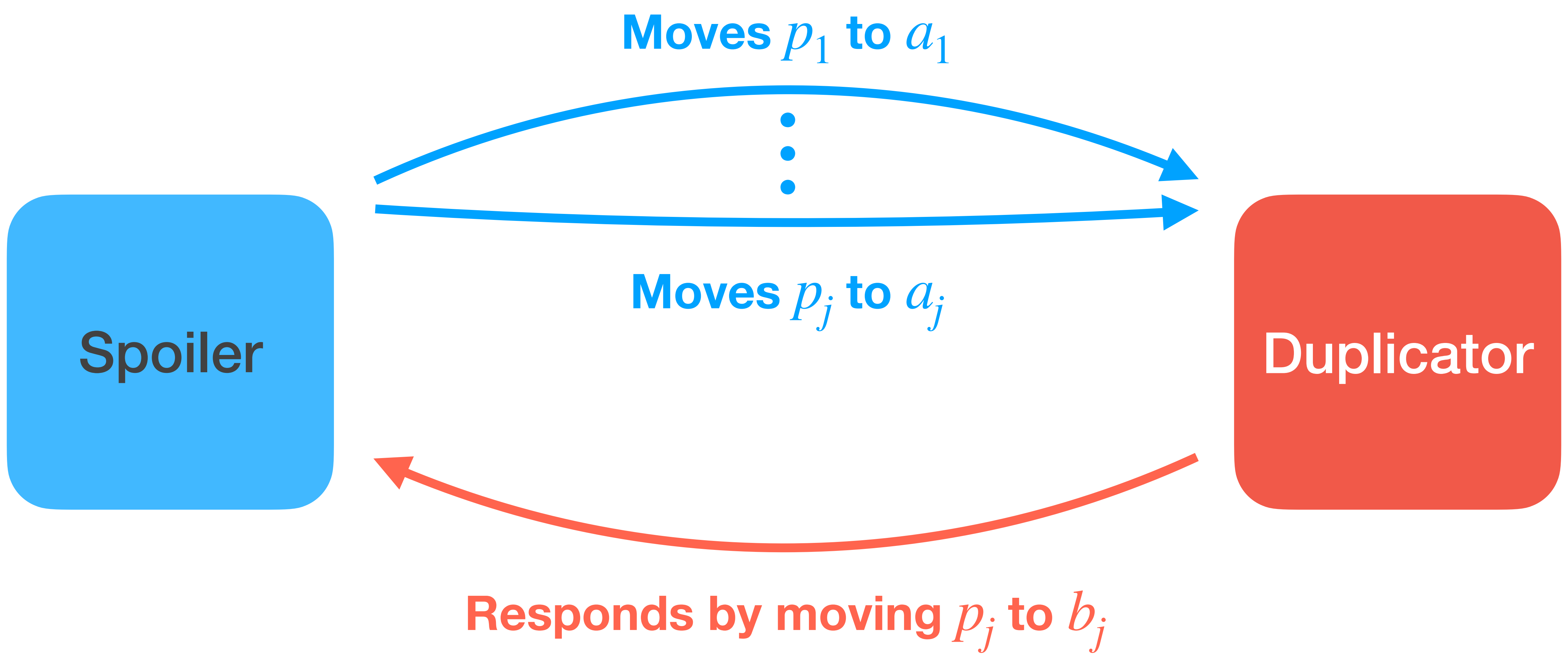

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$$[(p_1, a_1)] \mapsto b_1$$

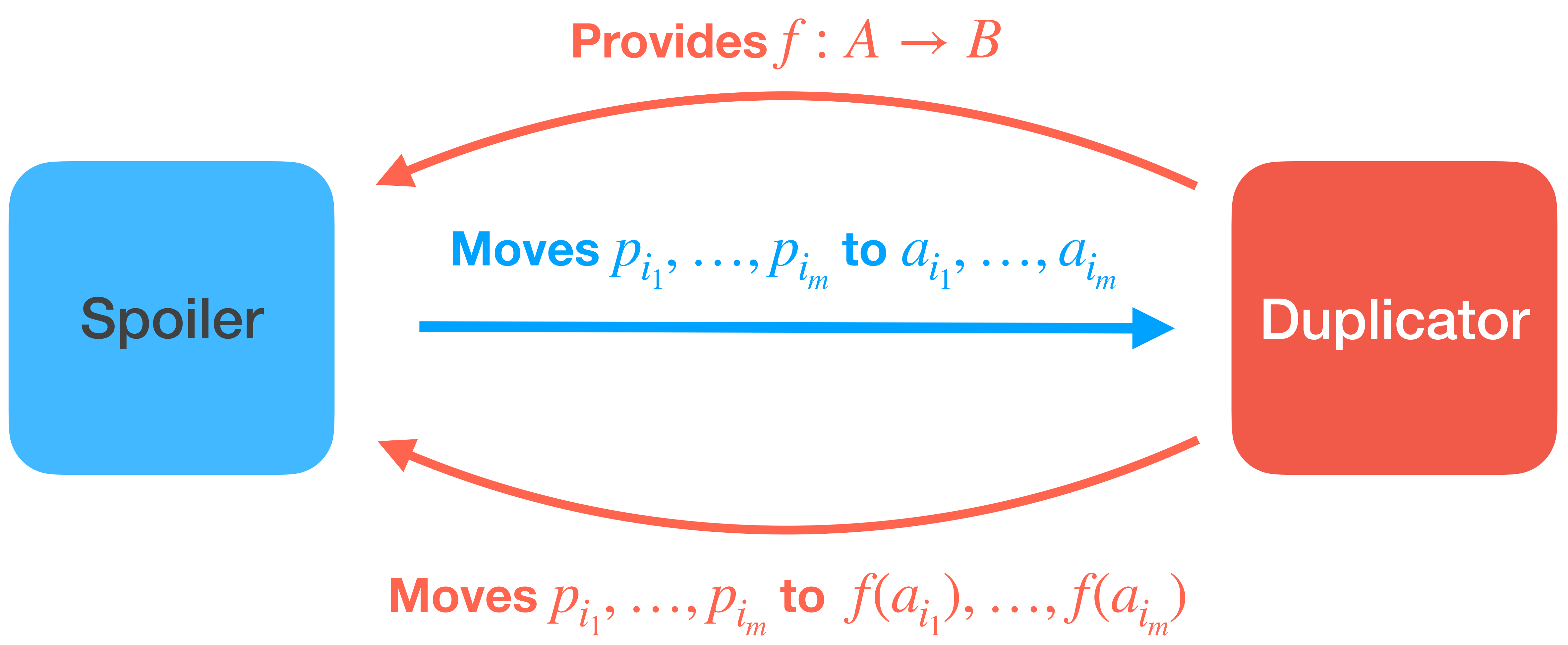
$$\vdots$$

$$[(p_1, a_1), \ldots, (p_j, a_j)] \mapsto b_j$$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $+\mathsf{Fun}^k_n(\mathscr{A}, \mathscr{B})$

A homomorphism $\mathbb{G}_{n,k}\mathscr{A} \to \mathscr{B}$



???

Lemma 20 (Ó C. & Dawar, 2021)

Duplicator has a winning strategy for $+\mathsf{Fun}^k_n(\mathscr{A}, \mathscr{B})$ if and only if she has an "n-consistent" winning strategy for $\exists\mathsf{Peb}_k(\mathscr{A}, \mathscr{B})$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's "n-consistent" strategy for $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$

A "special" homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

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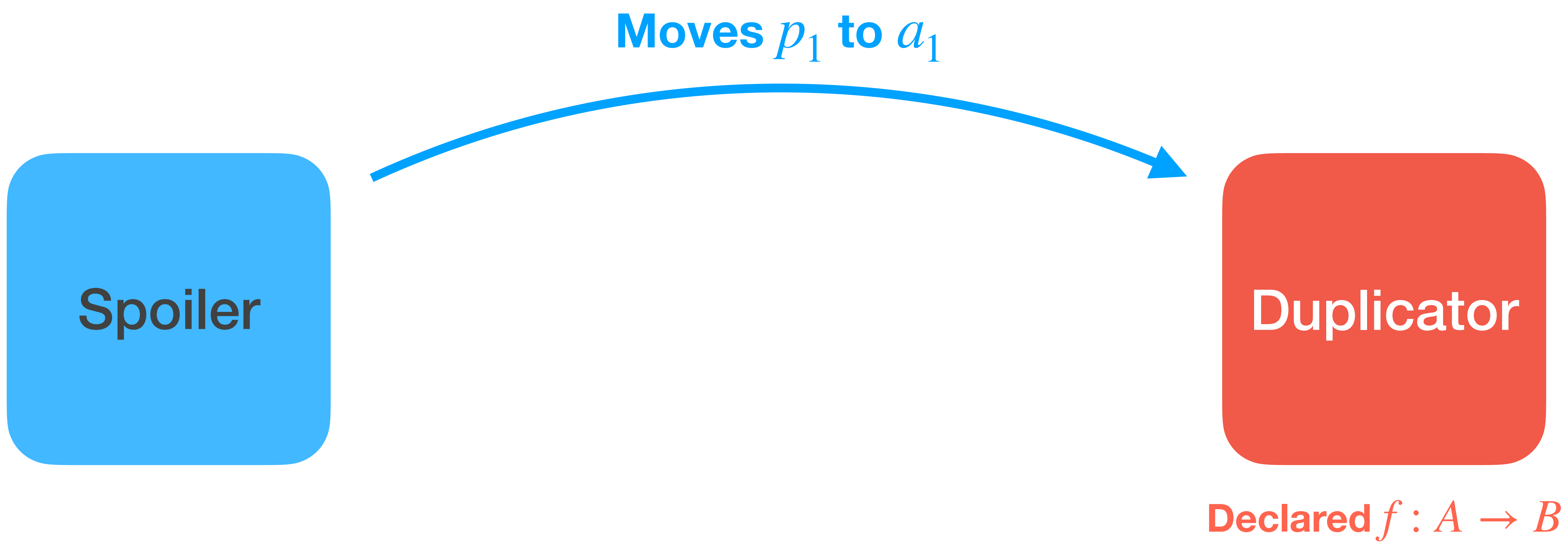

$[(p_1, a_1)] \quad \mapsto$

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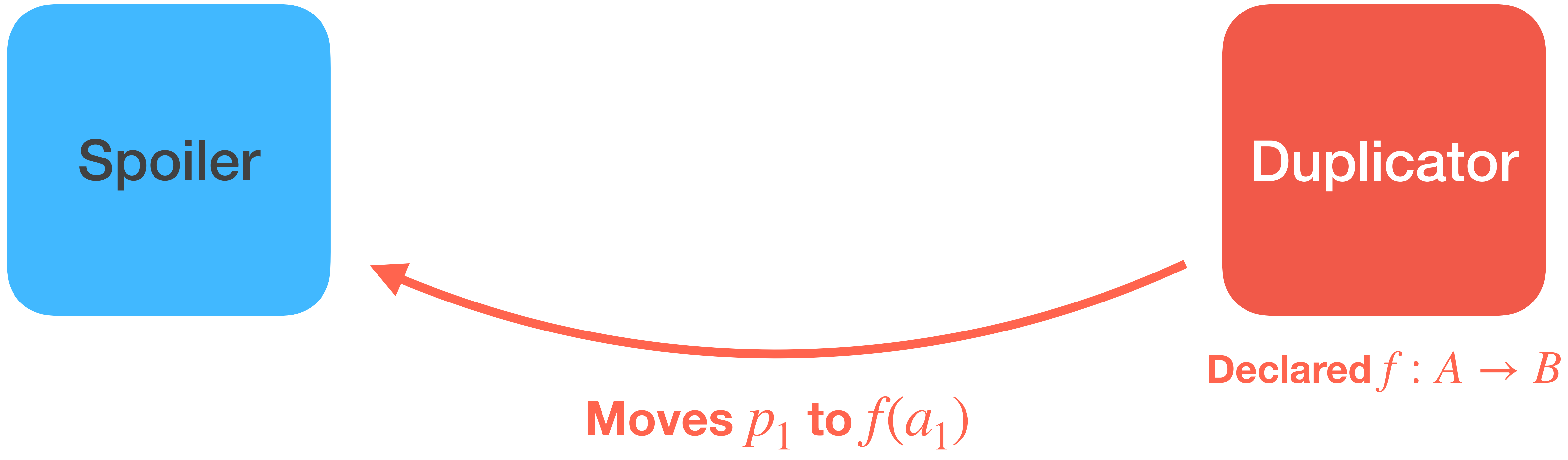

$[(p_1, a_1)] \mapsto f(a_1)$

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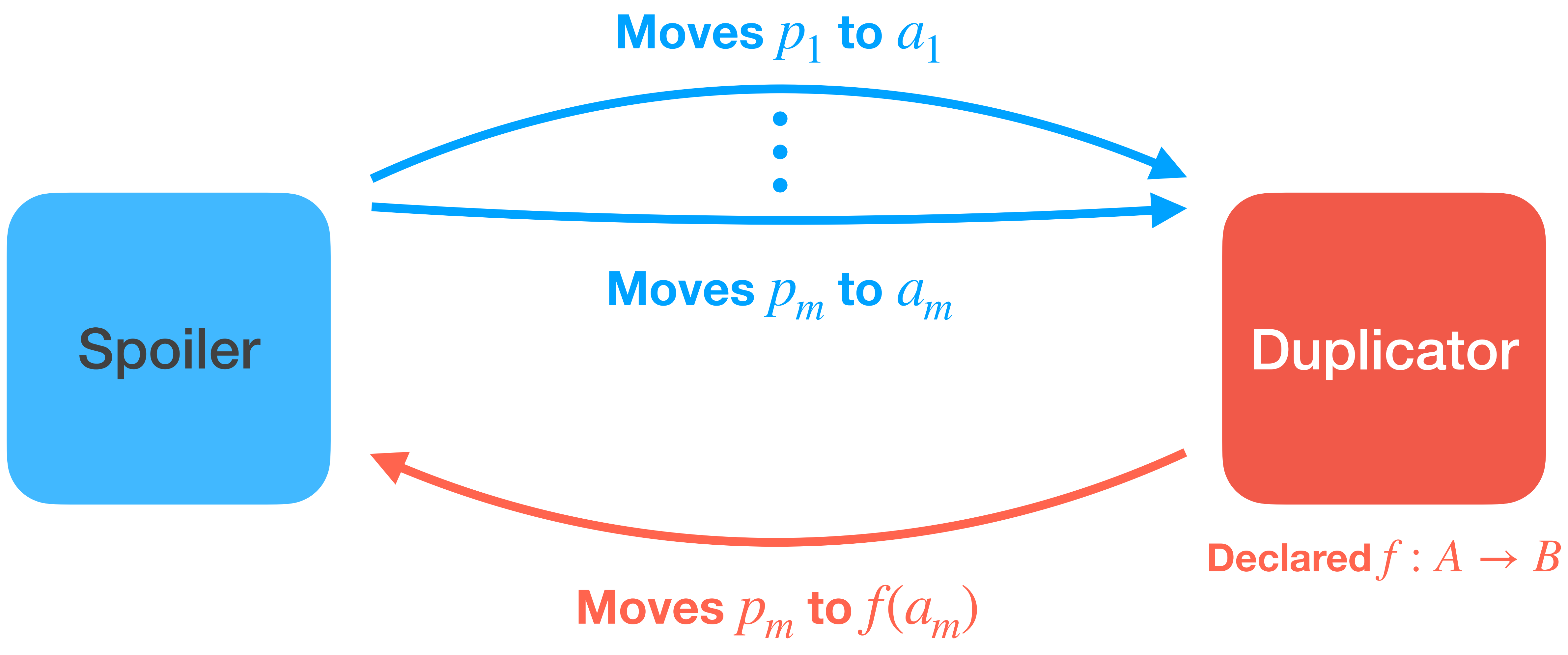


$$[(p_1, a_1)] \quad \mapsto \quad f(a_1)$$

$$\vdots$$

$$[(p_1, a_1), \ldots (p_m, a_m)] \quad \mapsto \quad f(a_m)$$

Game continues with Duplicator declaring a new f after Spoiler moves n pebbles (or earlier if Spoiler repeats a pebble).

$\exists$ an equiv. rel. $\approx_n$ s.t. homomorphism $\mathbb{P}_k\mathscr{A}/\approx_n \to \mathscr{B} \iff$ n-consistent strategy for Duplicator in $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$

$\iff$ strategy for Duplicator in $+\mathrm{Fun}^k_n(\mathscr{A}, \mathscr{B})$

Consequences of this new comonad

$$\mathbb{G}_{n,k}\mathscr{A} = \mathbb{P}_k\mathscr{A}/\approx_n$$

Kleisli Category $\mathscr{K}(\mathbb{G}_{n,k})$

$\mathbb{G}_{n,k}\mathscr{A} \to \mathscr{B} \iff$ Duplicator has a winning strategy for $+\mathrm{Fun}_n^k(\mathscr{A}, \mathscr{B})$

$\mathscr{A} \cong_{\mathscr{K}(\mathbb{G}_{n,k})} \mathscr{B} \iff$ Duplicator has a winning strategy for $\mathrm{Bij}_n^k(\mathscr{A}, \mathscr{B})$

Coalgebras

$\alpha : \mathscr{A} \to \mathbb{G}_{n,k}\mathscr{A} \iff \mathscr{A}$ has an extended tree decomposition of width k and arity n

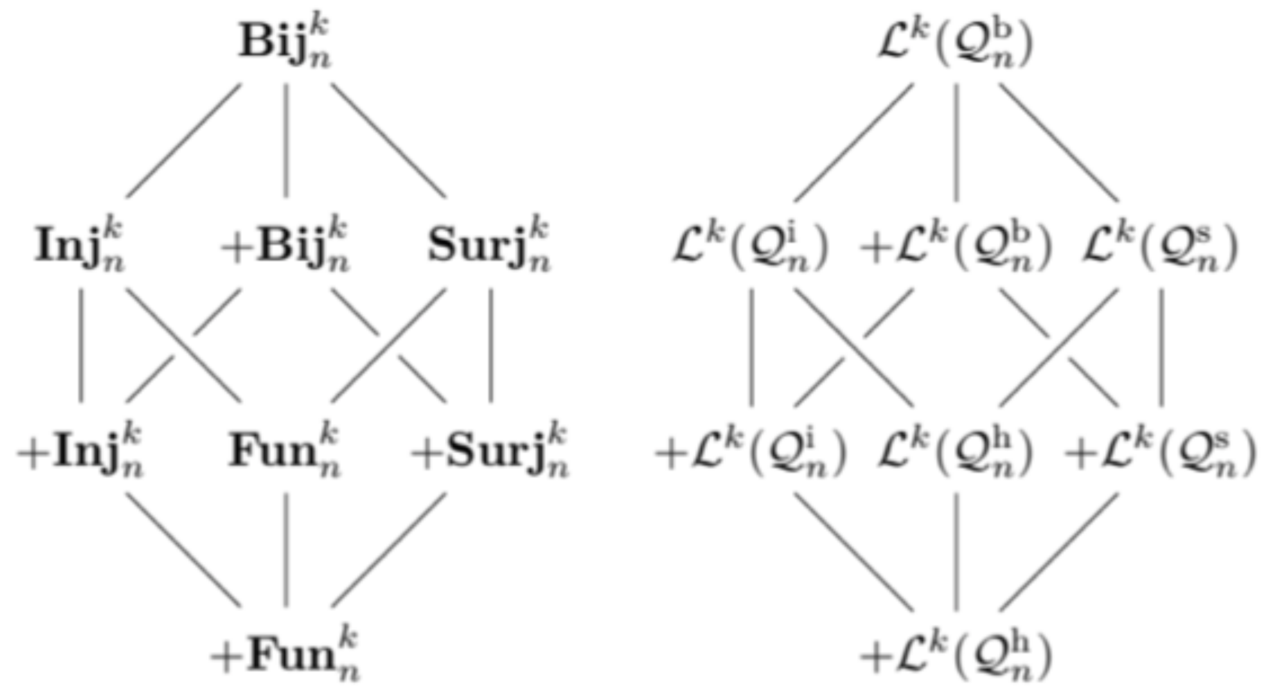
Conclusions & Future Directions

A much clearer understanding of the relation between quantifiers and the Kleisli Category of game comonads

$\to_{\mathscr{K}}$ is $\Rrightarrow_{\exists^{+}\mathscr{L}}$

and

$\cong_{\mathscr{K}}$ is $\Rrightarrow_{\mathscr{L}(\exists^{\geq m})}$



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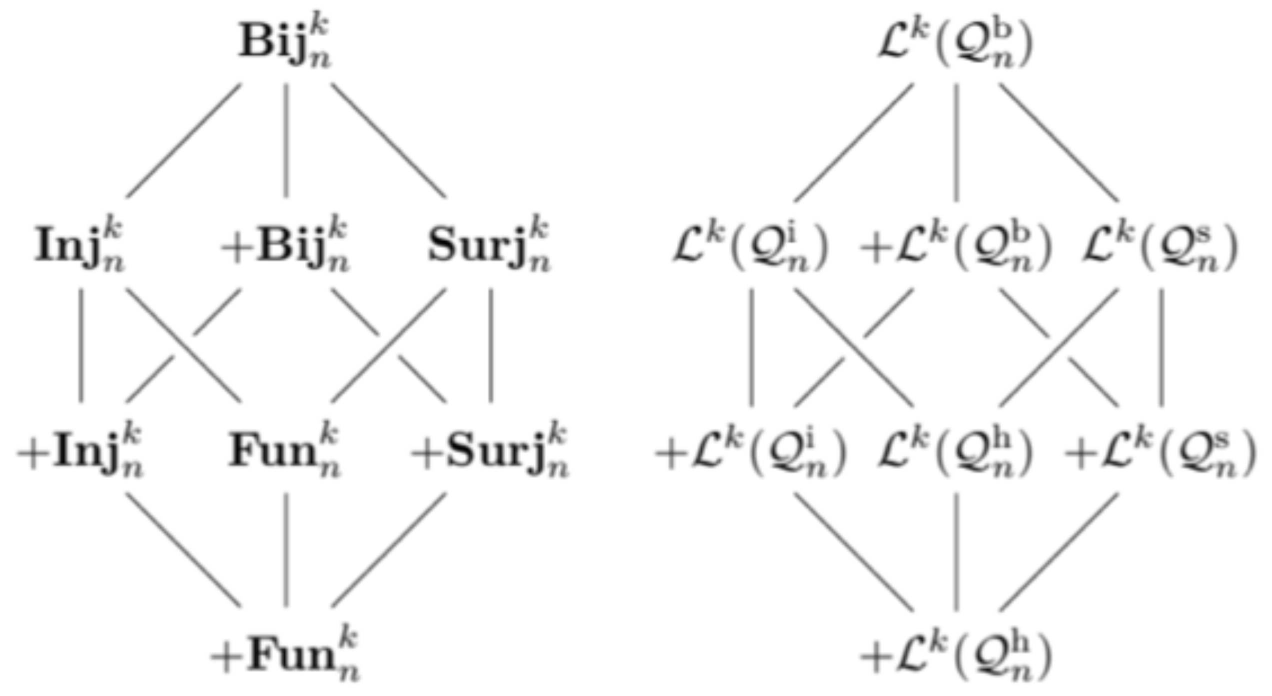

A method for constructing new games and new game comonads from old ones.
Can we turn more game theoretic translations into category theory?

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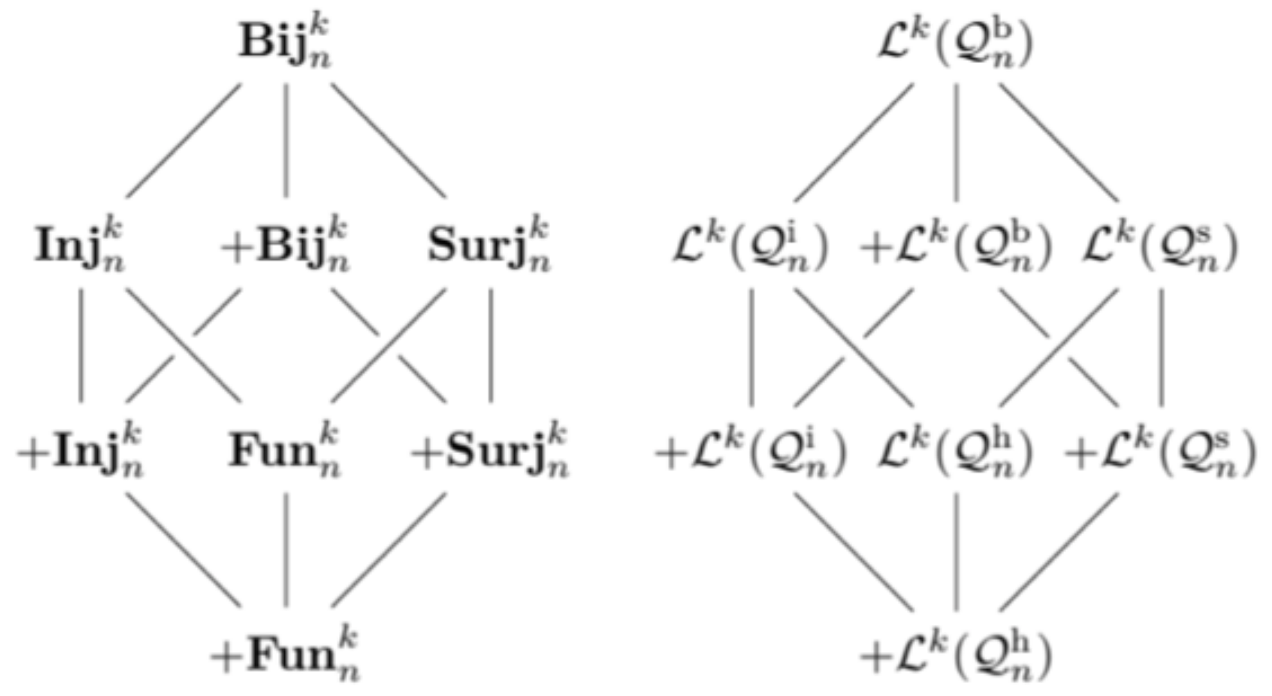

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Can we turn more game theoretic translations into category theory?

$$\mathbb{G}_{n,k}\mathscr{A} = \mathbb{P}_k\mathscr{A} / \approx_n$$

Some of the candidate logics for P (e.g. rank logic) are defined using classes of generalised quantifiers.
Can techniques from this work help us to make new comonads for these logics?