# Cohomology & pebble games

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I Pebble games, forth systems, algorithms.

II Sheaves in quantum foundations: cohomology $\Rightarrow$ contextuality.

III Cohomology & algorithms

IV Forth systems as presheaves

} other way around

## I Pebble games

(a) The $\exists$ k pebble game from A to B is a model comparison game giving rise to the following equivalent relaxations of $A \to B$.

① Duplicator has a winning strategy in the game.

② $A \Rrightarrow_{\exists^+ \mathcal{L}^k} B$

③ $\exists\ f : \mathbb{P}_k A \to B$ an $\mathcal{I}$-morphism in $\mathcal{K}(\mathbb{P}_k)$.

(④ The k-consistency alg accepts $(A, B)$)

⑤ $\exists\ \emptyset \neq S \subseteq \mathrm{Hom}_k(A,B)$ a forth system in $\mathrm{Hom}_k(A,B)$.

$S \subseteq \mathrm{Hom}_k(A,B)$ is a forth system if it is

(a) non-empty (b) down-ward closed (c) $\forall f \in S$, $\mathrm{Forth}(s,S)$: if $|\mathrm{dom}(f)| < k$

eg $U \subset \mathrm{dom}(f)$ $f \in S \Rightarrow f|_U \in S$

$\forall a \in A\ \exists b \in B$ s.t. $f \cup \{(a,b)\} \in S$.

The k-consistency algorithm for CSP can be thought of as the following algorithm

Input: $(A,B)$

Procedure: Set $S_0 := \mathrm{Hom}_k(A,B)$ enter following loop

~~for all $s \in S_{i-1}$~~

set $S_i := S_{i-1}$

for all $s \in S_{i-1}$

if $\neg \mathrm{Forth}(s, S_{i-1})$

remove all $s' \geq s$ from $S_i$.

if $S_i = S_{i-1}$ terminate accept

if $S_i = \emptyset$ terminate reject

repeat.

This is clearly in PTIME for fixed $k$. If we accept we say $A \to_k B$

(b) The $k$ pebble bijection game from $A$ to $B$ is a model comparison game giving rise to the following equivalent relaxations of $A \cong B$

1. Duplicator has a winning strategy in the game
2. $A \equiv_{\mathcal{C}^k} B$
3. $A$ and $B$ are isomorphic in $\mathcal{K}(\mathbb{P}_k)$
4. (The $k$-Weisfeiler-Leman* alg accepts $(A,B)$)
5. $\exists\, \emptyset \neq S \subseteq \mathrm{Isom}_k(A,B)$ a bijective forth system in $\mathrm{Isom}_k(A,B)$

* - usually people call this $(k-1)$-WL because of the original formulation in terms of $(k-1)$-tuples.

$S \subseteq \mathrm{Isom}_k(A,B)$ is a **bijective forth system** if it is

(a) non-empty (b) down-ward closed (c) $\forall f \in S$, $\mathrm{BijForth}(s,S)$: if $|\mathrm{dom}(f)| < k$ $\exists b_f : A \to B$ a bijection s.t. $\forall a \in A$ $f \cup \{(a, b_f(a))\} \in S$.

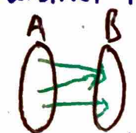
The **k-WL algorithm** for the isomorphism problem can be thought of as the following algorithm

**Input**: $(A,B)$

**Procedure**: Set $S_0 := \mathrm{Isom}_k(A,B)$, enter the following loop

set $S_i := S_{i-1}$
for all $s \in S_{i-1}$
  if $\neg \mathrm{BijForth}(s, S_{i-1})$ $\longleftarrow$
    remove all $s' \supseteq s$ from $S_i$
if $S_i = S_{i-1}$ terminate accept
if $S_i = \emptyset$ terminate reject
repeat.

For any $s \in S_{i-1}$, $|\mathrm{dom}(s)| < k$ construct the bipartite graph




$E = \{(a,b) \mid s \cup \{(a,b)\} \in S_{i-1}\}$

then existence of bijection in $\mathrm{BijForth}(s, S_{i-1})$ is a perfect matching in this graph.

This is clearly in PTIME for fixed $k$. If we accept we write $A \equiv_k B$.

(c). Both k-consistency and k-WL are known not to capture all PTIME $\to$ or $\cong$ problems (e.g. $\overbrace{\text{affine/Mal'cev/WNU CSPs}}^{\to}$ & $\overbrace{\text{CFI/Lichter properties}}^{\cong}$)

However, the "next steps" in $\to$ and $\cong$ worlds diverge (Bulatov/Zhuk $\to$; Babai or $\equiv_{IM}$ $\cong$)

**Main Question**: Is there a unified approach (maybe even comonadic) to these more powerful PTIME algorithms?

# II Sheaves in quantum foundations

(a) An S-valued presheaf on $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{op} \longrightarrow S$$

(this is especially important when $S = \mathbf{Set}$ or better when $S$ is abelian, see cohomology)

(for this talk it isn't important what a sheaf is)

Let's now see an important set of examples from quantum foundations. (Abramsky & Brandenburger, Abramsky, Barbosa, Kishida, Lal & Mansfield)

A measurement scenario is a triple $\langle X, \mathcal{C}, O \rangle$ where

- $X$ is a set (of quantum operators)
- $\mathcal{C}$ is a subset of $\mathcal{P}(X)$ (of ~~set~~ "contexts" of commuting operators) such that $\bigcup \mathcal{C} = X$. (this is a category under $\subseteq$)
- $O$ is a set (of outcomes)

The sheaf of outcomes of such a measurement scenario is the set valued presheaf $\mathcal{E} : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ given by

(i) $\mathcal{E}(U) = O^U$ (outcomes over each context)

(ii) $\mathcal{E}(U' \subseteq U) = (-)|_{U'}$ (the restriction operator $O^U \to O^{U'}$).

A possibilistic empirical model on $\langle X, \mathcal{C}, O \rangle$ is a subpresheaf $S$ of $\mathcal{E}$ such that

- $\forall C \in \mathcal{C} \quad S(C) \neq \emptyset$
- $S(U' \subseteq U) : S(U) \to S(U')$ is surjective (this is the condition of being a flasque presheaf.)
- If there exists a compatible family
  - $\{ s_C \in S(C) \}_{C \in \mathcal{C}}$ s.t. $\forall C, C' \quad s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$

  then there exists a global section
  - $g \in O^X$ s.t. $\forall U \in \mathcal{C} \quad g|_U \in S(U)$. (the reverse implication always holds.)

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(b) Contextuality

An empirical model $S$ of $(X, \mathcal{C}, O)$ is (logically) contextual if $S$ has no global section, write $LC(S)$.

$S$ is contextual at $s \in S(U)$ if there is no global section ~~such~~ $g \in O^X$ of $S$ such that $g|_U = s$, write $LC(s, S)$.

As we will see in section IV finding such global sections is hard in general so we need an easier condition for determining $LC(S)$ and $LC(s, S)$.

For this a related, AbGp-valued, presheaf $\mathcal{F}_S$ is considered which is defined as

- $\mathcal{F}_S(U) = \mathbb{Z}S(U) = \left\{ \sum_{s \in S(U)} \alpha_s s \;\middle|\; \alpha_s \in \mathbb{Z} \right\}$ (i.e. the $\mathbb{Z}$-module generated by elems of $S(U)$)
- $\mathcal{F}_S(U' \subseteq U)\left(\sum \alpha_s s\right) = \sum \alpha_s s|_{U'}$. (where $s|_{U'}$ is the formal variable corresponding to the elem $s|_{U'} \in S(U')$)

The construction of the Čech cohomology of $X$ valued in $\mathcal{F}_S$, $\check{H}^0(X, \mathcal{F}_S)$ inspires the following definitions

- $S$ is <u>cohomologically contextual at $s \in S(U)$</u> if there is no family $\{r_C \in \mathcal{F}_S(C)\}_{C \in \mathcal{C}}$ such that
  - $r_U = s$
  - $\forall C, C' \quad r_C|_{C \cap C'} = r_{C'}|_{C \cap C'}$

  write $CC(s, S)$.

$S$ is <u>cohomologically strongly contextual</u> if for all $s \in S(U)$ $CC(s, S)$

write $CSC(S)$.

<u>Important results</u>

- For any $S, s$ $\quad CC(s, S) \Rightarrow LC(s, S)$ & $CSC(S) \Rightarrow LC(S)$.
- If $O$ has the structure of any ring and $S$ "defines a theory over this ring which is unsatisfiable" then $CSC(S)$.

# III Forth systems as presheaves

(a) We now see how this framework applies to the systems of sets from finite model theory introduced in Section I.

For any structures $A, B$ and $k \in \mathbb{N}$ consider the triple $\langle A, A^{\leq k}, B \rangle$ where $A^{\leq k} = \{ U \in \mathcal{P}(A) \mid |U| \leq k \}$.

This is a valid measurement scenario, as in Section II.

The sheaf of outcomes $\mathcal{E}_k$ is simply that of partial functions from $A$ to $B$ with domains $\leq k$.

We consider $\mathcal{H}_k$ the subpresheaf of partial homomorphisms where

$$\mathcal{H}_k(U) = \{ f \in \mathrm{Hom}_k(A,B) \mid \mathrm{dom}(f) = U \}$$

and the restriction operator is well-defined.

Any downwards closed $S \subseteq \mathrm{Hom}_k(A,B)$ defines a subpresheaf of $\mathcal{H}_k$ where $S(U) = \{ f \in S \mid \mathrm{dom}(f) = U \}$ and dw closure guarantees that restriction is well-defined.

We now see our first dividend from this approach

~~Theo~~ Result: $S \subseteq \mathrm{Hom}_k(A,B)$ is a forth system iff
$S \subseteq \mathcal{H}_k$ is a flasque presheaf.

For bijective forth systems a similar analysis can be done in much the same way.

Let $\mathcal{I}_k \subseteq \mathcal{E}_k$ be the subpresheaf of partial isomorphisms where.

$$\tilde{\mathcal{I}}_k(U) = \{ f \in \mathrm{Isom}_k(A,B) \mid \mathrm{dom}(f) = U \}.$$

Again any dw-closed $S \subseteq \mathrm{Isom}_k(A,B)$ defines a subpresheaf of $\hat{\mathcal{I}}_k$, but the bijective forth property needs something more than flasque (what?)

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(b) Contextuality in forth systems

For $S \subseteq \mathcal{H}_k$ (or $S \subseteq \mathcal{I}_k$), a global section is (by II) $g \in \mathrm{Hom}(A,B)$ (or $g \in \mathrm{Isom}(A,B)$) such that $\forall U$ $g|_U \in S(U)$.

This means that $S$ has a global section if there is a full homomorphism (or isomorphism) $g : A \to B$ which locally "looks like" $S$.

~~So if $S$ is flasque but logically consistent~~

So if $\mathcal{H}_k$ has flasque subpresheafs but is logically contextual this says that $A \to_k B$ but $A \not\to B$.

# IV Cohomology & algorithms

(a) Recall that for structures $A, B$

- the $k$-consistency algorithm constructs the maximal dw-closed $\bar{S} \subseteq \mathrm{Hom}_k(A,B)$ s.t. $\forall s \in \bar{S}$ $\mathrm{Forth}(s, \bar{S})$
- the $k$-WL algorithm constructs the maximal dw-closed $\bar{\bar{S}} \subseteq \mathrm{Isom}_k(A,B)$ s.t. $\forall s \in \bar{\bar{S}}$ $\mathrm{BijForth}(s, \bar{S})$.

B. In language of Section II, we would like to continue by removing any $s \in S$ s.t. $LC(s, S)$. (any local section of a hom/isom will remain in any case)

However $LC(s,S)$ is equivalent to finding a hom/isom $g: A \to B$ s.t. $g|_u = s$ which is as hard in general as $CSP(A,B)$ / $Iso(A,B)$.

<u>Idea</u> Instead remove $s \in S$ s.t. $CC(s, S)$.

By definition, this involves checking $\mathbb{Z}$ linear equations

- $$\sum_{s' \in S(u)} \alpha_{s'} s' = s$$
- $\forall C, C'$ $$\sum_{s_1 \in S(C)} \alpha_{s_1} s_1|_{C \cap C'} = \sum_{s_2 \in S(C')} \alpha_{s_2} s_2|_{C \cap C'}$$

For fixed $k$ this is a polynomial* number of equations in a polynomial number of variables

* - in $|A| \cdot |B|$.

(b) This suggests new algorithms

<u>Cohomological k-consistency</u>

<u>Input</u> : $(A, B)$

<u>Procedure</u>: Run k-consistency $(A,B)$ and <u>reject</u> or let $S_0 := \overline{S}$
and enter the following loop:

set $S_i := S_{i-1}$
for all $s \in S_{i-1}$:
  if $CC(s, S_i)$ } ← this is done by solving $\mathbb{Z}$-linear equations on last page.
    remove all $s' \geq s$ from $S_i$
if $S_i = S_{i-1}$ terminate <u>accept</u>
if $S_i = \emptyset$ terminate <u>reject</u>
repeat.

As $\mathbb{Z}$-linear equations are solvable in PTIME, so is this algorithm.

If it succeeds we write $A \xrightarrow{\mathbb{Z}}_k B$.

<u>Cohomological k-WL</u>

<u>Input</u> $(A,B)$

<u>Procedure</u>: Run k-WL on $(A,B)$ and reject or have $S_0 := \overline{\overline{S}}$, and enter loop

set $S_i := S_{i-1}$
for all $s \in S_{i-1}$
  if $CC(s, S_{i-1})$ or $CC(s, S_{i-1}^{-1})$ or $\neg BijForth(s, S_{i-1})$
    remove all $s' \geq s$ from $S_i$.
if $S_i = S_{i-1}$ terminate <u>accept</u>
if $S_i = \emptyset$ terminate <u>reject</u>
repeat.

Also PTIME & write $A \equiv^{\mathbb{Z}}_k B$.

# Results

- $A \to_k^{\mathbb{Z}} B \iff \exists \emptyset \neq S \subseteq Hom_k(A,B)$ dw-closed s.t. $\forall s \in S \ \neg CC(s,S)$
- $A \equiv_k^{\mathbb{Z}} B \iff \exists \emptyset \neq S \subseteq Isom_k(A,B)$ dw-closed s.t. $\forall s \in S \ \neg CC(s,S)$ $\wedge$ $BijForth(s,S)$
- $A \to_k^{\mathbb{Z}} B \to_k^{\mathbb{Z}} C \Rightarrow A \to_k^{\mathbb{Z}} B$
- $A \equiv_k^{\mathbb{Z}} B \Rightarrow A \to_k^{\mathbb{Z}} B \wedge B \to_k^{\mathbb{Z}} A$
- $\Phi : R(\sigma) \to R(\tau)$ a $\ell^{\ell}$-interpretation $\exists n$ $A \equiv_{n\ell}^{\mathbb{Z}} B \Rightarrow \Phi(A) \equiv_n^{\mathbb{Z}} \Phi(B)$.
- Power of these algorithms

## CSP

| Alg / Problem | $\to_k$ | $\to_k^{\mathbb{Z}}$ | Bulatov/Zhuk |
|---|---|---|---|
| Bounded width | ✓ | ✓ | ✓ |
| Affine | ✗ | ✓ | ✓ |
| Mal'cev | ✗ | ? | ✓ |
| WNU | ✗ | ? | ✓ |

| | $\equiv_k$ | $\equiv_{IM}$ | $\equiv_k^{\mathbb{Z}}$ |
|---|---|---|---|
| CFI | ✗ | ✓ | ✓ |
| Lichter | ✗ | ✗ | ✓ |
| | | | |

# Future directions

1. Which CSPs does $\to_k^{\mathbb{Z}}$ solve? (for fixed $k$)
2. ~~What~~ Are there examples $\{(A_k, B_k)\}_{k \in \mathbb{N}}$ s.t. $A_k \not\cong B_k$ but $A_k \equiv_k^{\mathbb{Z}} B_k$?
3. Is there a comonad for $(\to_k^{\mathbb{Z}}, \equiv_k^{\mathbb{Z}})$?