Categorical tools for Descriptive Complexity Theory

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Outline

- Finite Models & Computation
- Lower bounds in Descriptive Complexity
- Comonads & possible connections to DST.

Expressiveness & Computational Complexity

Let σ be a finite relational signature

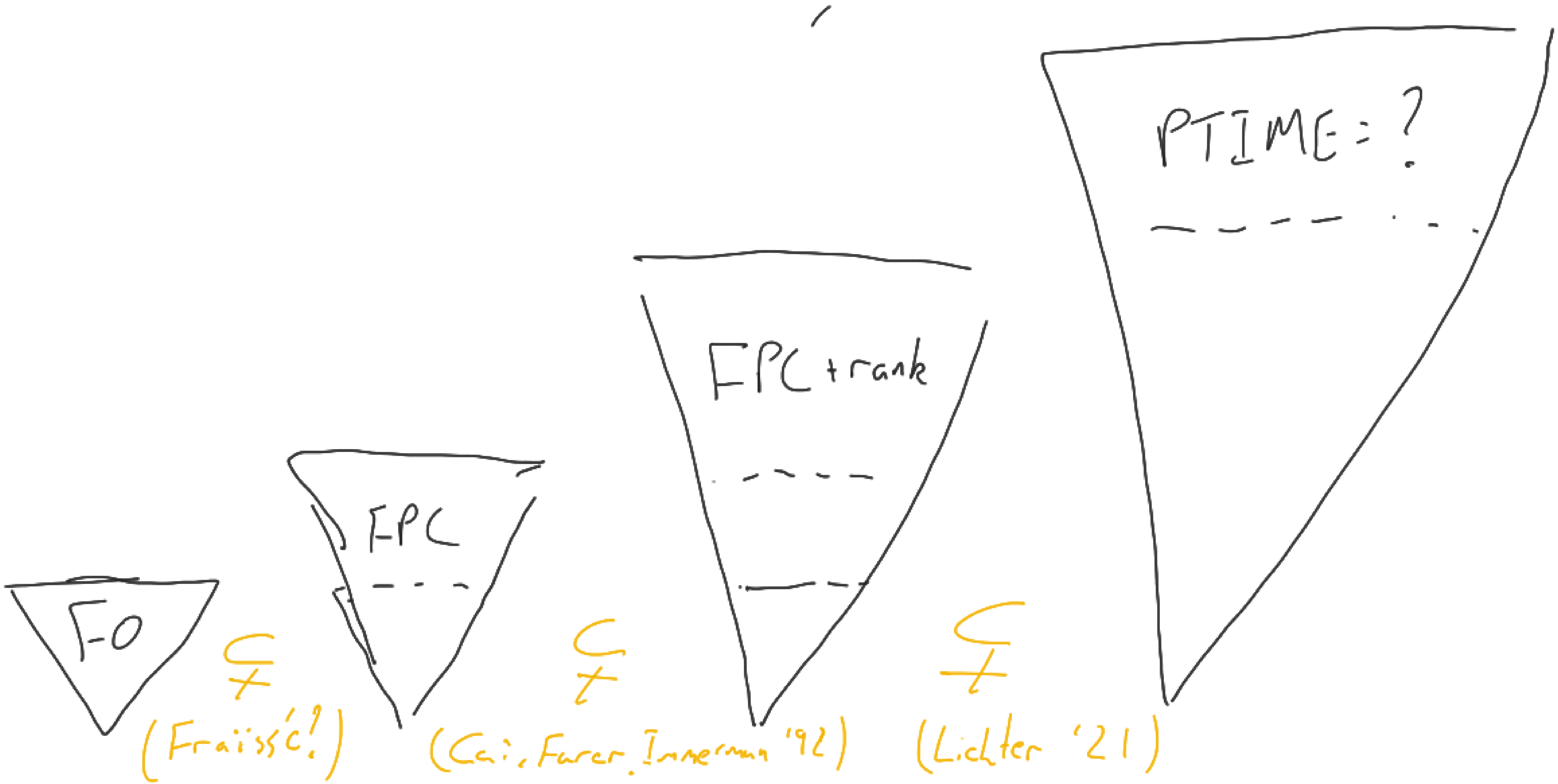
$\mathcal{R}(\sigma)$ category of (finite) σ-structures & homomorphisms

[Fagin '73] ("Logic can capture computational complexity")

$\mathcal{C} \subseteq \mathcal{R}(\sigma)$ a class of finite structures

$$\mathcal{C} \text{ in } NP \iff \mathcal{C} = Mod(\varphi) \text{ for } \varphi \in \exists SO.$$

Is there a $\mathcal{L}$ capturing P?



How do we prove these lower bounds?

Problem: On finite models, $A \equiv_{FO} B \iff A \cong B$

$A \Rrightarrow_{\exists + FO} B \iff A \to B$!

So to show C not definable in $\mathcal{L}$ need

- $\mathcal{L} = \bigcup_k \mathcal{L}^k$

- $\{A_k\}, \{B_k\}$ s.t. $A_k \in C$ $B_k \notin C$

  & $A_k \equiv_{\mathcal{L}^k} B_k$. ← "Duplicator" winning some game.

Categorical tools for these hierarchies [Abramsky, Dawar, Wang '17]

<u>Idea</u>: Represent game using endofunctor $g : R(\sigma) \to R(\sigma)$

g-winning strategy for Duplicator is hom $gA \longrightarrow B$.

<u>Results</u>: Many such endofunctors are <u>comonads</u> where

$$\left\{ \begin{array}{l} \text{coKleisli maps = one-way games} \\ \text{coKleisli isoms = two-way games} \\ \text{coalgebras = decompositions} \end{array} \right\}$$

Examples

Logics	Games	Comonads
$FO+C \subseteq \bigcup_n \mathcal{L}_n(\#)$	Ehrenfeucht-Fraïssé	$\mathbb{E}_n$ [Abramsky + Shah '18]
$FPC \subseteq \bigcup_k \mathcal{L}^k_{\infty\omega}(\#)$	Pebble games	$\mathbb{P}_k$ [Abramsky, Dawar, Wang '17]
$FO(Q_n) \subseteq \bigcup_k \mathcal{L}^k_{\infty\omega}(Q_n)$	Hella games	$\mathbb{H}_{n,k}$ [Ó C., Dawar '21]
$FO(\mathbb{Z}) \subseteq \bigcup_k \mathcal{L}^k_{\infty\omega}(\mathbb{Z})$	$\mathbb{Z}$-linear games	$\mathbb{Z}_k$ [Abramsky, Ó C., Dawar f/c]

Why am I here?

Game Comonads	Computable Endofunctors
• g-winning maps capture "logical relaxations" of homomorphisms in $R(\sigma)$	• d-continuous maps capture "topological relaxations" of cts maps in category of represented spaces
... and other interesting logical data g-isom, g-back-and-forth, g-decompositions	... and other interesting top data d-measurable, d-admissable, d-Hausdorff

Question:

- Is there a connection between these pictures?
  - In particular
    - Can we recast game comonads in terms of rep spaces?
    - Can game comonads help understand computable monads?
    - Is there a "synthetic" descriptive complexity theory?