Game Comonads & Generalised Quantifiers

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joint with *Anuj Dawar*

LCC 2020

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- Our Work and the $\mathbb{G}_{n,k}$ Comonad

Logic and Games

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"If you give me an x I can give you a y such that for any z you pick, x, y and z are related by R"

This is a game and winning strategies for me are (constructive) proofs that

$$\mathcal{U} \models \phi$$

for whatever universe we're allowed to choose x, y and z from.

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Spoiler-Duplicator Games

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One important such genre of game is the *Spoiler-Duplicator* game played by two players on two structures $\mathcal{A}$ and $\mathcal{B}$ which share a relational structure.

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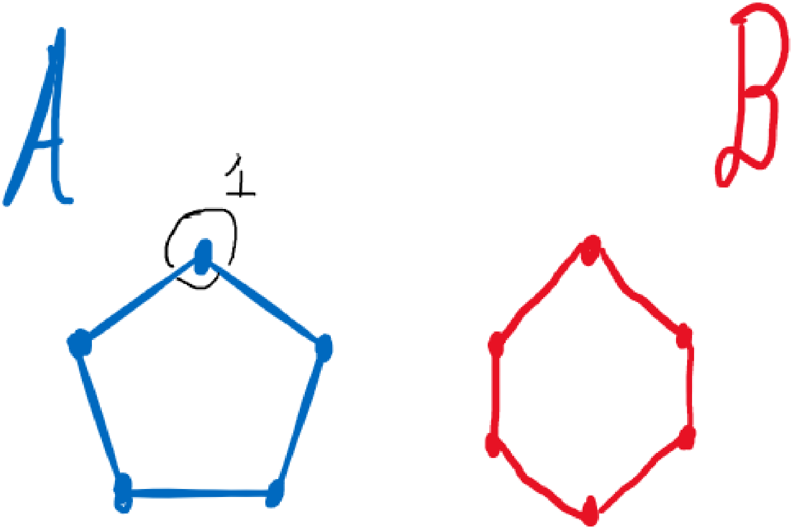
One important such genre of game is the *Spoiler-Duplicator* game played by two players on two structures $\mathcal{A}$ and $\mathcal{B}$ which share a relational structure.

Duplicator wants to show that $\mathcal{A}$ and $\mathcal{B}$ are related in some way, Spoiler wants to prove Duplicator wrong.

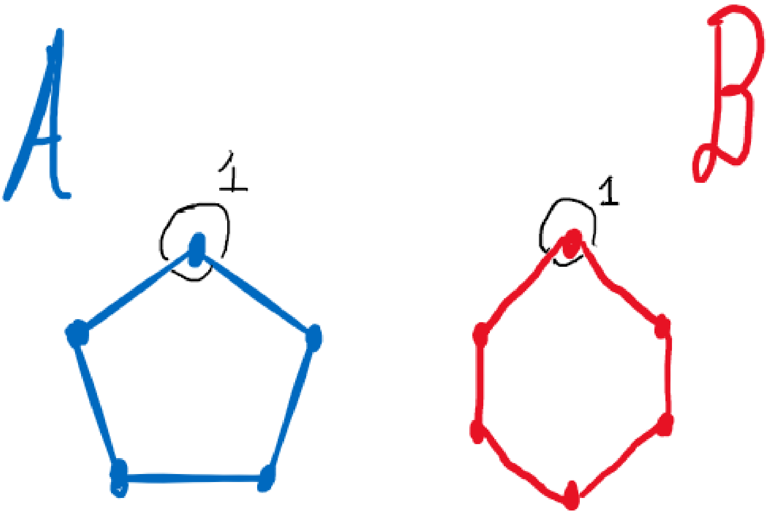
Generic “one-way” Spoiler-Duplicator game

Spoiler marks some $a_1 \in \mathcal{A}$

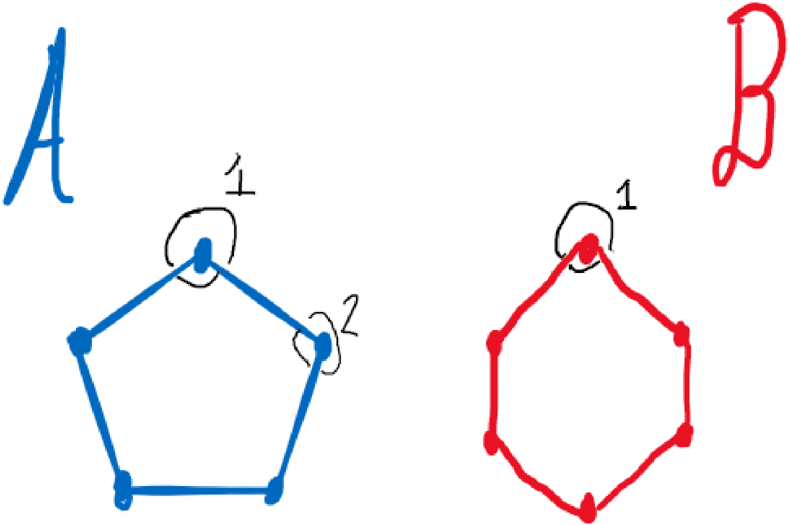


Generic "one-way" Spoiler-Duplicator game

Duplicator responds with some $b_1 \in \mathcal{B}$



Spoiler marks some $a_2 \in \mathcal{A}$



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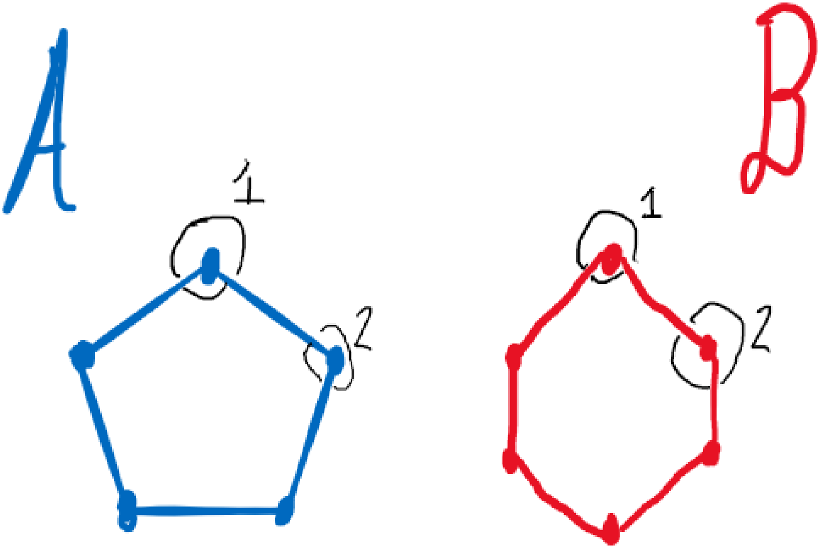
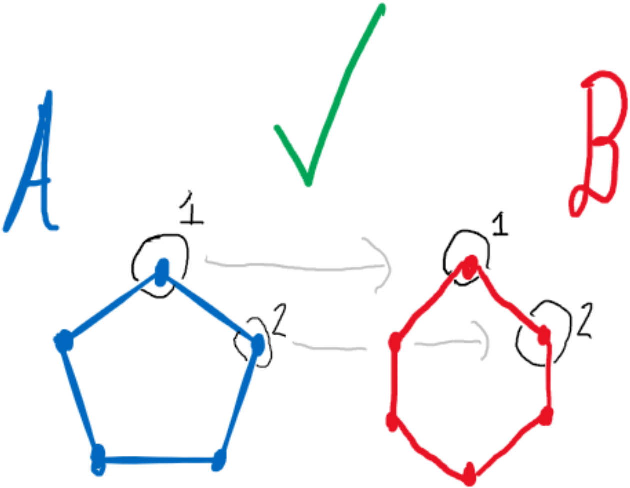


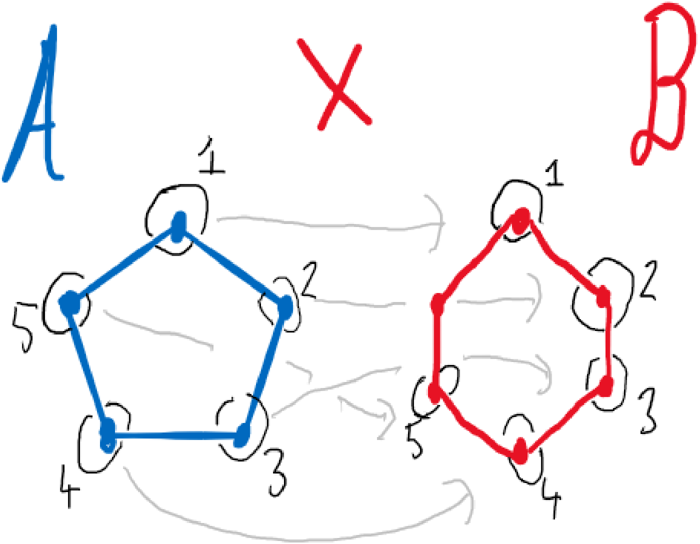
Generic "one-way" Spoiler-Duplicator game

After every round we check that for any related tuples marked in $\mathcal{A}$ are preserved by Duplicators response.



Generic “one-way” Spoiler-Duplicator game

If this test fails, Spoiler wins.



Generic “one-way” Spoiler-Duplicator game

Duplicator wins if they can play forever and strategies are constructive proofs that $\mathcal{A} \Rrightarrow_L \mathcal{B}$ i.e. $\forall \phi \in L$, $\mathcal{A} \models \phi \implies \mathcal{B} \models \phi$.

In this case of the above game $L = \exists^+ \mathcal{L}_\infty$.

The flexibility of Spoiler-Duplicator Games

The beauty of Spoiler-Duplicator games is their flexibility.

If we make it harder for Spoiler to win, L will be more restrictive.

If we make it harder for Duplicator to win, L will be more expressive.

Changing the rules

Name	Rules	Logic
$\exists^+\mathbf{EF}_n$	Play for n rounds	$\exists^+\mathbf{FO}_n$
$\exists^+\mathbf{Peb}^k$	Limit to k pebbles	$\exists^+\mathcal{L}^k_\infty$

Table: Limits on spoiler $\iff$ syntactic restrictions

Name	Rules	Logic
$\mathbf{Bij}^k_1$	Duplicator responds with a bijection[1]	$\mathcal{L}^k_\infty + \#$
$\mathbf{Bij}^k_n$	Same bijection for n rounds	$\mathcal{L}^k_\infty + \mathcal{Q}_\mathbf{n}$

Table: Limits on Duplicator $\iff$ syntactic expansions

[1] and the test we use is for partial isomorphism rather than homomorphism

Hella's Game $\mathbf{Bij}_n^k$

We focus now on the last of these games with the most restrictions on Duplicator. This game, due to Hella[2], is called the n-bijective k-pebble and each round is played as follows:

- Duplicator now plays first in any round by providing a bijection $f : A \to B$.
- Spoiler moves (up to) n pebbles in $\mathcal{A}$ to new positions $a_1, \ldots a_m$ and the corresponding pebbles in $\mathcal{B}$ move to $f(a_1), \ldots f(a_m)$.
- Spoiler wins if the pebbles positions don't define a partial **isomorphism**.

[2] [Hel89]

This is very **hard** for Duplicator to win!

So the logic must be very **expressive**.

As proved by Hella[3], this logic is $\mathcal{L}^k_\infty(\mathcal{Q}_n)$

[3] [Hel89]

$\mathcal{L}^k_\infty(\mathcal{Q}_n)$: generalised quantifiers

Let σ, τ be relational signatures.
An *interpretation* from σ to τ in a logic L is a family of formulas

$$\Psi(\mathbf{x}) = (\psi_R(\mathbf{x_R}))_{R \in \tau}$$

in $L[\sigma]$ where $|\mathbf{x_R}| = \text{arity}(R)$. This defines a function $\Psi : \mathcal{R}(\sigma) \to \mathcal{R}(\tau)$

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(actually a functor!)

$\mathcal{L}^k_\infty(\mathcal{Q}_n)$: generalised quantifiers

Let K be an isomorphism-closed class of τ-structures. Then the logic $\mathcal{L}^k_\infty(Q_K)[\sigma]$ is the logic which extends $\mathcal{L}_\infty$ by formulas of the form

$$\phi(\mathbf{y}) := Q_K\mathbf{x}.\ \Psi(\mathbf{x}, \mathbf{y})$$

where for any fixed $\mathbf{a}$, $\Psi_{\mathbf{a}}(\mathbf{x}) := \Psi(\mathbf{x}, \mathbf{a})$ is an interpretation from σ to τ and $\mathcal{A}, \mathbf{a} \models \phi(\mathbf{y}) \iff \Psi_{\mathbf{a}}(\mathcal{A}) \in K$.

We say Q_K has *arity* n if $\max_{R\in\tau}(\text{arity}(R)) = n$.

$\mathcal{L}^k_\infty(\mathcal{Q}_n)$ is the infinitary FO extended by *all* n-ary quantifiers

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- ... *even undecideable ones!*

However,

Hella, 1996 [Hel96]

For all $n, k \in \mathbb{N}$,

$$\mathrm{PTIME} \not\subset \mathcal{L}^k_\infty(\mathcal{Q}_n)$$

Case of $n = 1$

In the $n = 1$ case, this game and logic lead a double life!

Kolaitis & Väänänen [KV95]

$$\mathcal{L}^k_\infty(\mathcal{Q}_1) \equiv \mathcal{L}^k_\infty(\exists^{\geq m})$$

Additionally this equivalence corresponds to the approximation to isomorphism given by the $(k-1)$-Weisfeiler-Lehman algorithm.

Game comonads

Observation/Question

Games like Hella's approximate isomorphism while games like the "one-way" pebble games approximate homomorphism.
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Answer *(to this and more):*

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Answer *(to this and more):* Game comonads!

Abramsky, Dawar & Wang's pebbling comonad [5]

Idea: To create from $\mathcal{A}$ a structure $\mathbb{P}_k\mathcal{A}$ on the set of *histories of Spoiler moves* $(A \times [k])^+$

Results:

- This is a comonad with ϵ and δ adapted from that for non-empty lists
- In the Kleisli category $\mathcal{K}(\mathbb{P}_k)$:
  - Homomorphisms $\mathbb{P}_k\mathcal{A} \to \mathcal{B}$ are exactly the winning strategies for Duplicator in $\exists\mathbf{Peb}_k(\mathcal{A}, \mathcal{B})$
  - Isomorphisms $\mathbb{P}_k\mathcal{A} \cong \mathbb{P}_k\mathcal{B}$ are exactly the winning strategies for Duplicator in $\mathbf{Bij}_1^k(\mathcal{A}, \mathcal{B})$
- Coalgebras $\alpha : \mathcal{A} \to \mathbb{P}_k\mathcal{A}$ are exactly the tree decompositions of width $< k$ of $\mathcal{A}$

Abramsky and Shah[4] found similar constructions for other games: $\mathbb{E}_n$ for the n-round game, $\mathbb{M}_n$ for modal bisimulation.

[4] [AS18]
[5] [ADW17]

Our work on extending $\mathbb{P}_k$ upwards

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- What would coalgebras look like in this context?

We answer all of that and more in our work.

Overview of work

- New games
- New logics
- A new game comonad

New games: parametrised definition

We consider a game similar set up to the one-way k pebble game, except now each round proceeds as follows:

1. Duplicator provides $f : A \to B$ a homomorphism (injection, surjection or bijection respectively)
2. Spoiler moves up to n pebbles on $\mathcal{A}$ to positions $a'_1, \dots a'_m$, and respective pebbles on $\mathcal{B}$ move to $f(a'_1), \dots f(a'_m)$ on $\mathcal{B}$
3. Spoiler wins the *positive* game if the partial map defined by the pebbled positions fails to preserve positive atoms (i.e. is not a partial homomorphism) and the *full* game if it fails to preserve both positive and negative atoms.

New games

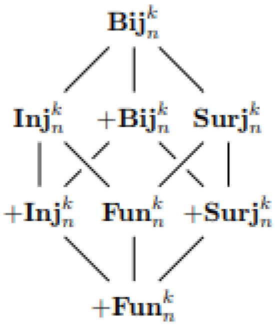


Figure 1: Hasse Diagram of Games

New classes of quantifiers

For K a class of σ-structures we say that K is *homomorphism-closed* (resp. injection, surjection, bijection-closed) if for all $f : \mathcal{A} \to \mathcal{B}$ a homomorphism (resp. injective, surjective, bijective homomorphism)

$$\mathcal{A} \in K \implies \mathcal{B} \in K$$

We also call the quantifiers Q_K built from such classes homomorphism-closed (resp. injection, surjection, bijection-closed)

In general

- $\mathcal{Q}_n^{\mathsf{h}}$ is the set hom-closed n-ary quantifiers
- $\mathcal{Q}_n^{\mathsf{i}}$ is the set inj-closed n-ary quantifiers
- $\mathcal{Q}_n^{\mathsf{h}}$ is the set surj-closed n-ary quantifiers
- $\mathcal{Q}_n^{\mathsf{h}}$ is the set bij-closed n-ary quantifiers

New logics

From these we build the new logics.
Note here that $\mathcal{L}^k$ is quantifier-free infinitary logic.

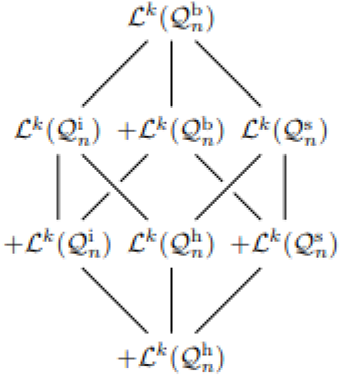
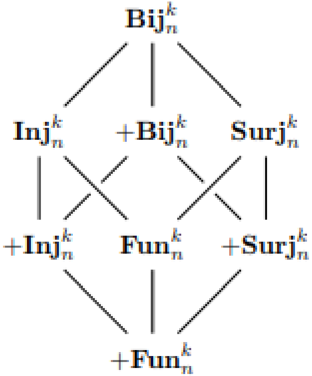
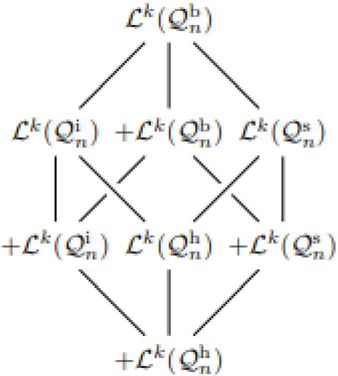


Figure 2: Hasse Diagram of Logics

We show in Theorem 3.20[6] that Duplicator winning strategies for these games correspond to the relation $\mathcal{A} \Rrightarrow_L \mathcal{B}$ for the these new logics.





[6] [CD20]

Case of $n = 1$

An interesting case that we have worked out is that of $n = 1$. Deepening our understanding of the games and logics involved in $\mathbb{P}_k$

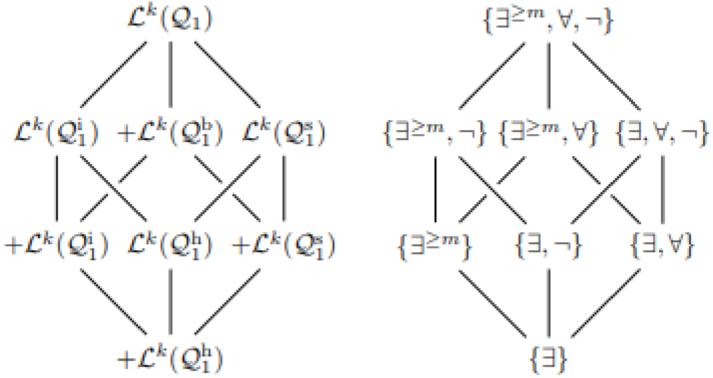


Figure 4: Logics with $n = 1$.

New comonad

We also build a comonad $\mathbb{G}_{n,k}$

- In the Kleisli category $\mathcal{K}(\mathbb{G}_{n,k})$:
  - Homomorphisms $\mathbb{G}_{n,k}\mathcal{A} \to \mathcal{B}$ are exactly the winning strategies for Duplicator in $+\textbf{Fun}^k_n(\mathcal{A}, \mathcal{B})$
  - Isomorphisms $\mathbb{G}_{n,k}\mathcal{A} \cong \mathbb{G}_{n,k}\mathcal{B}$ are exactly the winning strategies for Duplicator in $\textbf{Bij}^k_n(\mathcal{A}, \mathcal{B})$
  - winning strategies for $+\textbf{Inj}^k_n$, $+\textbf{Surj}^k_n$ and $+\textbf{Bij}^k_n$ are *branchi-injective, -surjective and -bijective* strategies for $+\textbf{Fun}^k_n$
- Coalgebras $\alpha : \mathcal{A} \to \mathbb{G}_{n,k}\mathcal{A}$ give us a new type of *extended tree decomposition* with width $\leq k$ and *arity* n

New comonad: how we build it

You could construct $\mathbb{G}_{n,k}\mathcal{A}$ directly by putting a relational structure on the set of histories of Spoiler moves (in this case $((A \times [k])^{\leq n})^* \times A$) as Abramsky et al. did for $\mathbb{P}_k$.

Our construction is instead indirect and shows how to construct $\mathbb{G}_{n,k}$ from $\mathbb{P}_k$ using a quotient. The key lemma is the following

Lemma 4.5 [CD20]

Duplicator has an n-*consistent* winning strategy in $\exists\mathbf{Peb}_k$ if, and only if, they have a winning stategy in $+\mathbf{Fun}_n^k$

The definition of n-consistent forces Duplicator to respond in the same way to Spoiler given certain histories which are deemed to be equivalent under some relation $\approx_n$ and we show that we can construct $\mathbb{G}_{n,k}\mathcal{A} := \mathbb{P}_k\mathcal{A}/\approx_n$ as a comonad.

- We have found the right notion of one-way game for Hella's n-bijection game. Can we similarly break up other games e.g IM-equivalence game?
- We have found a new way to construct new game comonads from old by translating strategies between games. What other comonads can we build like this?
- We found a systematic approach for understanding different classes of generalised quantifiers and how they interact. Can we do the same for more complex classes such as Dawar, Grädel and Pakusa's linear algebraic logic?[7]

[7] [DGP19]

Recap

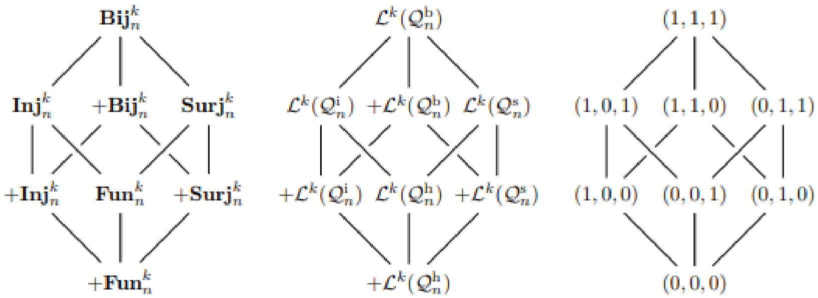


Figure 1 Hasse Diagrams of Games and Logics with Labels For Reference

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