Games & Comonads in Finite Model Theory

Swansea Logic Seminar Group

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Talk Outline

- A very rapid overview and motivation of my work
- A history of the connection between logic, games and complexity theory
- Game Comonads: a new framework for thinking about games in logic
- Oracles, generalised quantifiers and how these fit into game comonads

A quick introduction

Two Very Important Problems

	Constraint Satisfaction Problem	Graph Isomorphism Problem
	Is there a homomorphism?	Is there an isomorphism?
	$\mathscr{X} \to \mathscr{D}$?	$\mathscr{G} \cong \mathscr{H}$?
Uses	SAT-solvers, Sudoku querying databases	Verification, code optimisation, pattern recognition
Complexity	Either **P** or **NP-Complete** (Bulatov & Zhuk, 2018)	Not known to be **P** or **NP-Complete** Suspected "intermediate problem"
P-Time Approximations	k-local consistency algorithm	k-Weisfeiler-Lehman algorithm

Logic => Games => Game Comonads

Logic is the key to understanding these algorithms

k-local consistency algorithm

$$\mathscr{X} \to_k \mathscr{D} \quad \iff \quad \mathscr{X} \Rrightarrow_{\exists^+\mathscr{L}^k} \mathscr{D}\,?$$

Kolaitis & Vardi, 1992

k-Weisfeiler-Lehman algorithm

$$\mathscr{G} \cong_k \mathscr{H} \quad \iff \quad \mathscr{G} \equiv_{\mathscr{L}^k(\sharp)} \mathscr{H}$$

Immerman & Lander, 1990

Games are the key to understanding logic

$$\mathscr{X} \to_k \mathscr{D} \quad \Longleftrightarrow \quad \mathscr{X} \Rrightarrow_{\exists^+\mathscr{L}^k} \mathscr{D}\,?$$

Theorem (Kolaitis & Vardi, 1992)

"Duplicator" has a winning strategy for $\exists\text{Peb}_k(\mathscr{X}, \mathscr{D})$ if and only if $\mathscr{X} \Rrightarrow_{\exists^+\mathscr{L}^k} \mathscr{D}$

$\exists\text{Peb}_k(\mathscr{X}, \mathscr{D})$

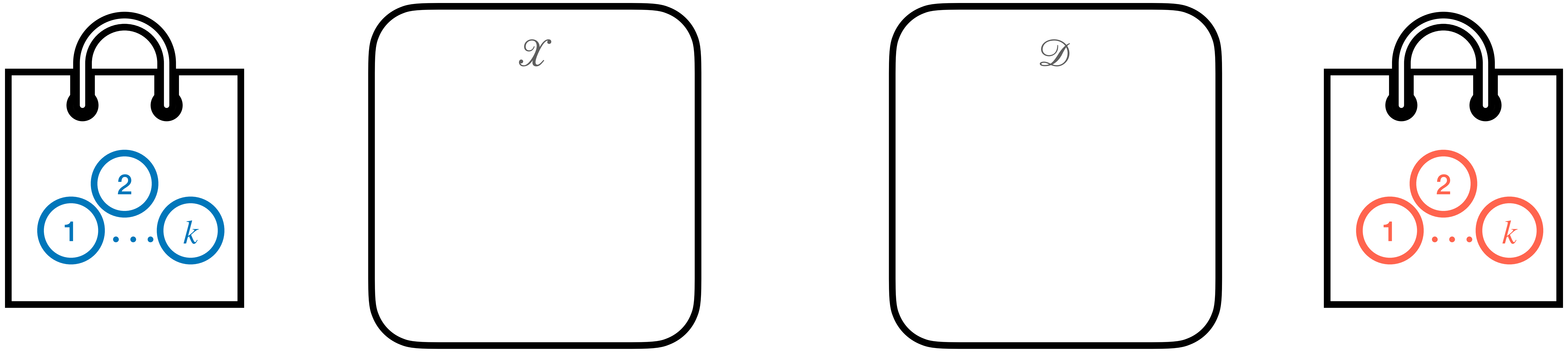
Spoiler

Spoiler wants to convince Duplicator that $\mathscr{X} \nrightarrow \mathscr{D}$

Duplicator wants to convince Spoiler that $\mathscr{X} \to \mathscr{D}$

…but they have limited access to $\mathscr{X}$ and $\mathscr{D}$

Duplicator



Games are the key to understanding logic

$$\mathscr{G} \cong_k \mathscr{H} \quad \Longleftrightarrow \quad \mathscr{G} \equiv_{\mathscr{L}^k(\sharp)} \mathscr{H}$$

Theorem (Hella, 1996)

Duplicator has a winning strategy for $\mathrm{Bij}_k(\mathscr{A}, \mathscr{B})$ if and only if $\mathscr{A} \equiv_{\mathscr{L}^k(\sharp)} \mathscr{B}$

$\mathrm{Bij}_k(\mathscr{A}, \mathscr{B})$

Spoiler

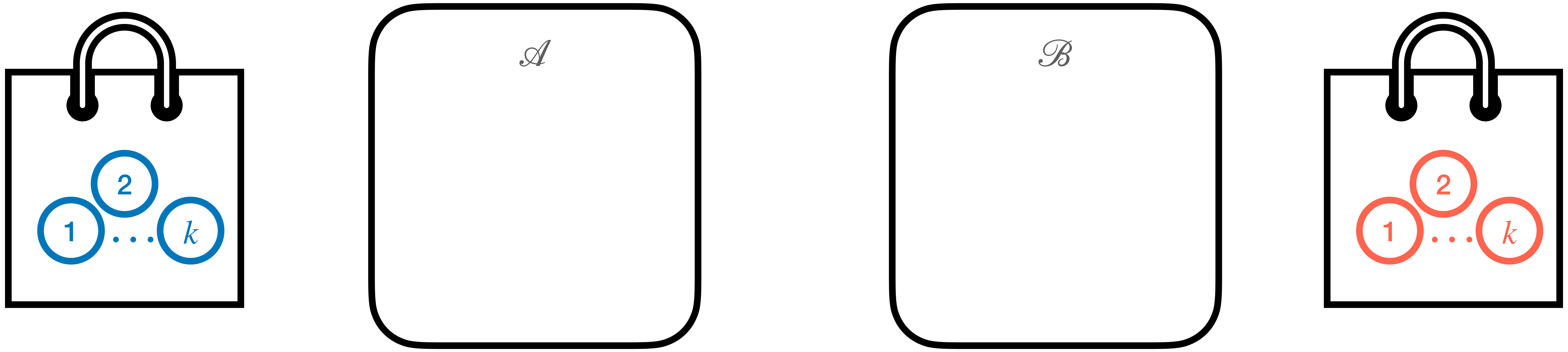
Spoiler wants to convince Duplicator that $\mathscr{A} \ncong \mathscr{B}$

Duplicator wants to convince Spoiler that $\mathscr{A} \ncong \mathscr{B}$

…they have limited access to $\mathscr{X}$ and $\mathscr{D}$

and Duplicator has to give a one-to-one map from A to B containing her moves

Duplicator



Game Comonads are the key to understanding these games

Research on Constraint Satisfaction

$\mathscr{X} \rightarrow_k \mathscr{D} \iff \mathscr{X} \Rrightarrow_{\exists^+\mathscr{L}^k} \mathscr{D} \iff$ Duplicator wins $\exists\mathrm{Peb}_k(\mathscr{X}, \mathscr{D})$

Research on Graph Isomorphism

$\mathscr{G} \cong_k \mathscr{H} \iff \mathscr{G} \equiv_{\mathscr{L}^k(\sharp)} \mathscr{H} \iff$ Duplicator wins $\mathrm{Bij}_k(\mathscr{G}, \mathscr{H})$

$$(\mathscr{R}(\sigma), \Rrightarrow_{\exists^+\mathscr{L}^k}, \equiv_{\mathscr{L}^k(\sharp)})$$

Missing Link?

"Pebbling" Comonad $\mathbb{P}_k$ (Abramsky, Dawar & Wang, 2018)

The history of logic, games & complexity

Finite Model Theory in One Slide

What models? $\mathscr{R}(\sigma)$ the class of relational structures over signature σ

$$\mathscr{A} = \langle A, (R^A)_{R\in\sigma}\rangle \qquad R^A \subset A^{\mathrm{ar}(R)}$$

Any logic over sigma $L[\sigma]$ comes with a semantics defining when $\mathscr{A} \vDash \phi$ for any $\phi \in L[\sigma]$

$f : \mathscr{A} \to \mathscr{B}$ is a homomorphism means that $\forall R \in \sigma,\ (a_1, \ldots, a_m) \in R^A \implies (f(a_1), \ldots f(a_m)) \in R^B$

$(\mathscr{R}(\sigma), \to)$ defines a category

Why finite?

For FO over all structures:

(Gödel's completeness theorem)

$\phi \in FO$ is consistent $\iff$ there is some model of ϕ

For FO over *finite* structures:

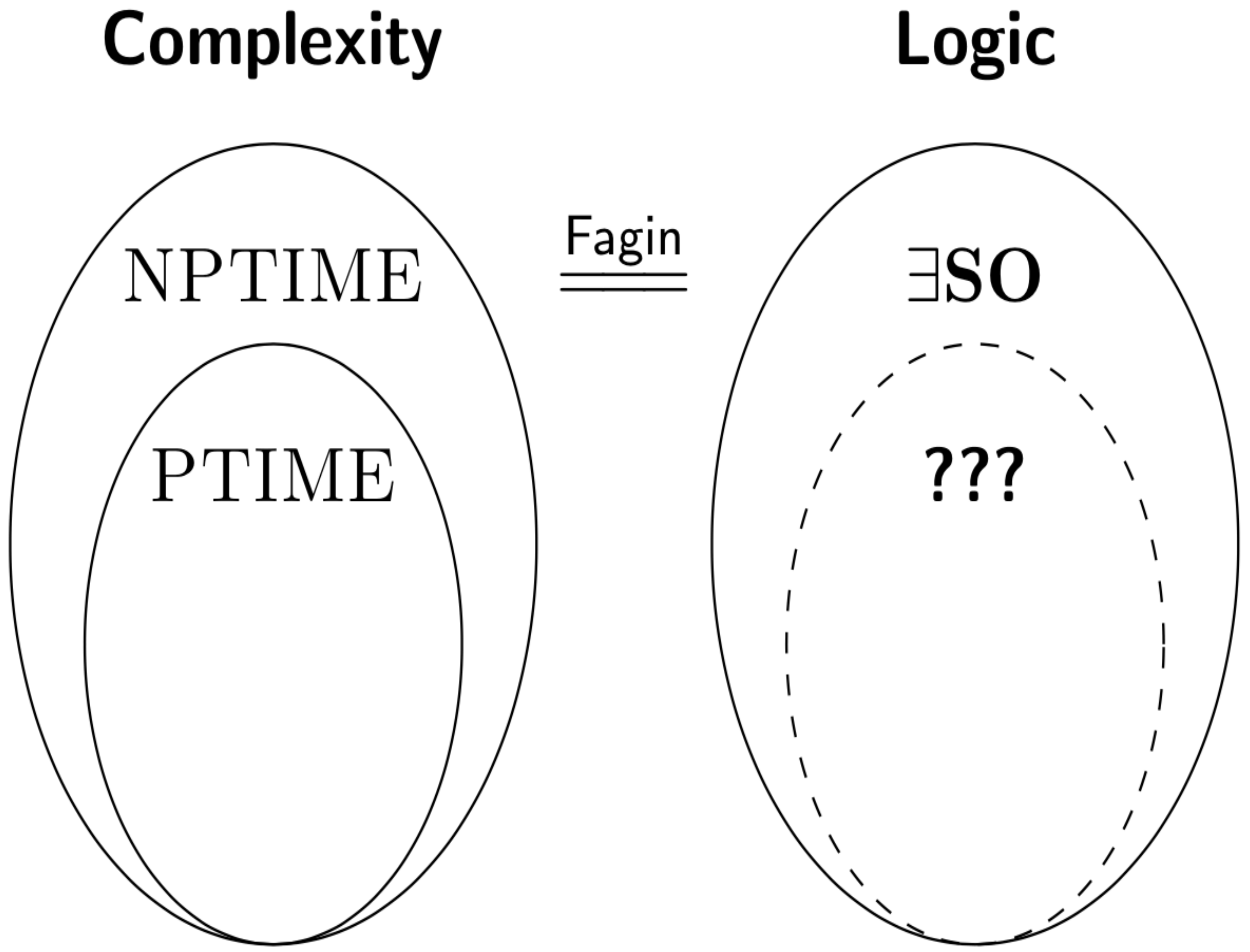
(Trakhtenbrot's Theorem)

$\{\phi \in FO \mid$ there is some finite model of $\phi\}$ is undecideable.

Descriptive Complexity

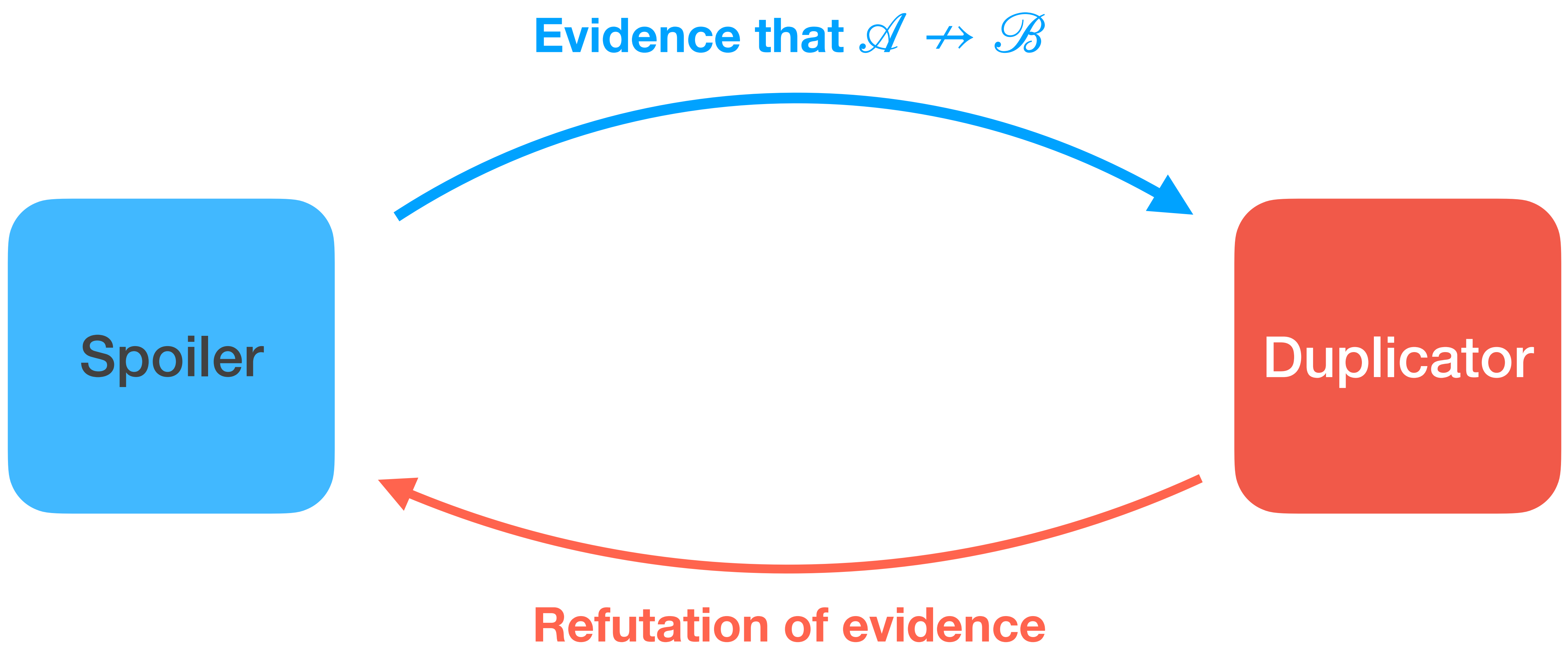
A quick tour

- (Fagin's Theorem, 1973)
  A class of finite structures is decidable in NP if and only if it is expressible in $\exists$SO
- (Gurevich's Conjecture, 1988)
  There is no equivalent logic for P
- (Cai, Furer, Immerman, 1992)
  $\mathscr{L}^k(\sharp) \neq$ PTIME, for any k.
- Candidate logics for P include rank logic, and choiceless polynomial time.



Games: a key tool for logic

Spoiler-Duplicator Games on relational structures $\mathscr{A}, \mathscr{B}$ over signature σ



<u>**One-way Games**</u>

Duplicator "wins" iff $\mathscr{A} \Rrightarrow_{\mathscr{L}} \mathscr{B}$

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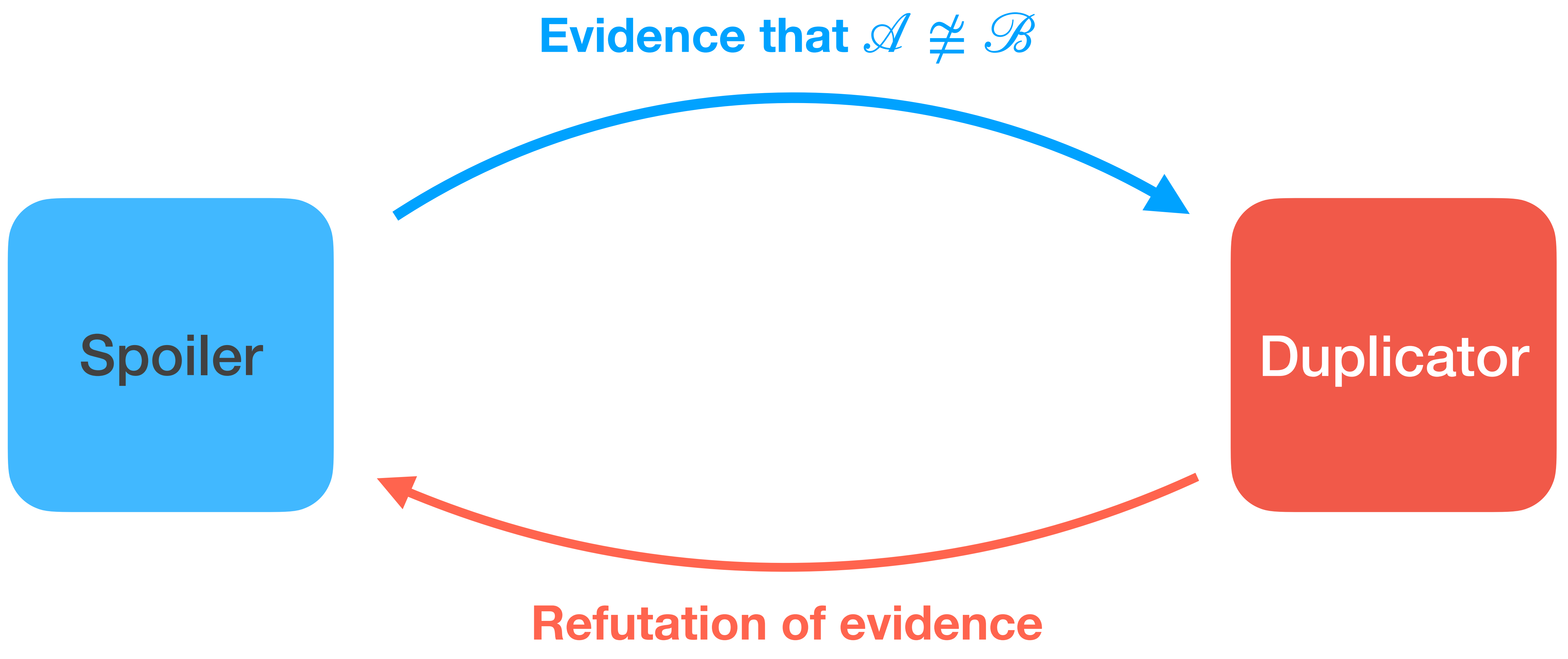


One-way Games

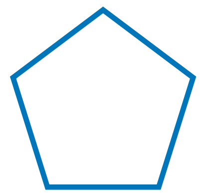
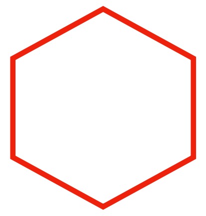
Duplicator "wins" iff $\mathscr{A} \Rrightarrow_{\mathscr{L}} \mathscr{B}$

Two-way Games

Duplicator "wins" iff $\mathscr{A} \equiv_{\mathscr{L}} \mathscr{B}$

The exact $\mathscr{L}$ depends on the rules of the game

Example of Spoiler-Duplicator Games

Ehrenfeucht-Fraïssé Game between  and 

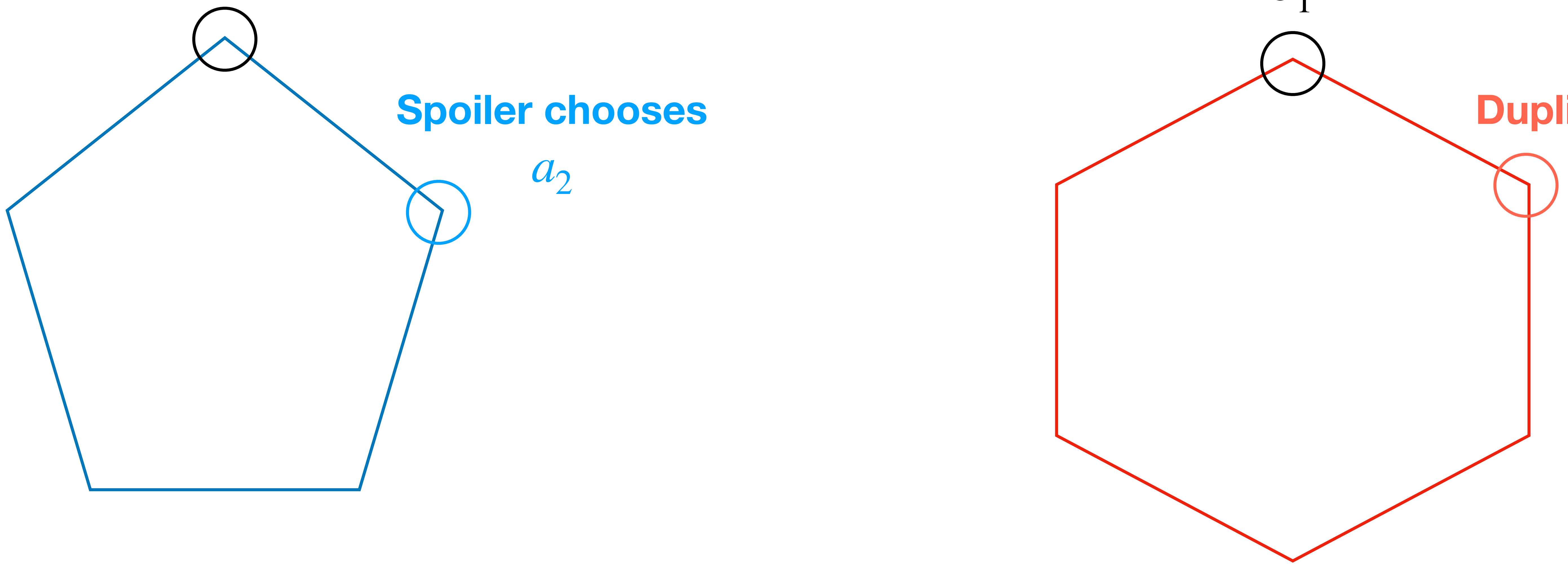
($\sigma = \{E\}$)

Round 1

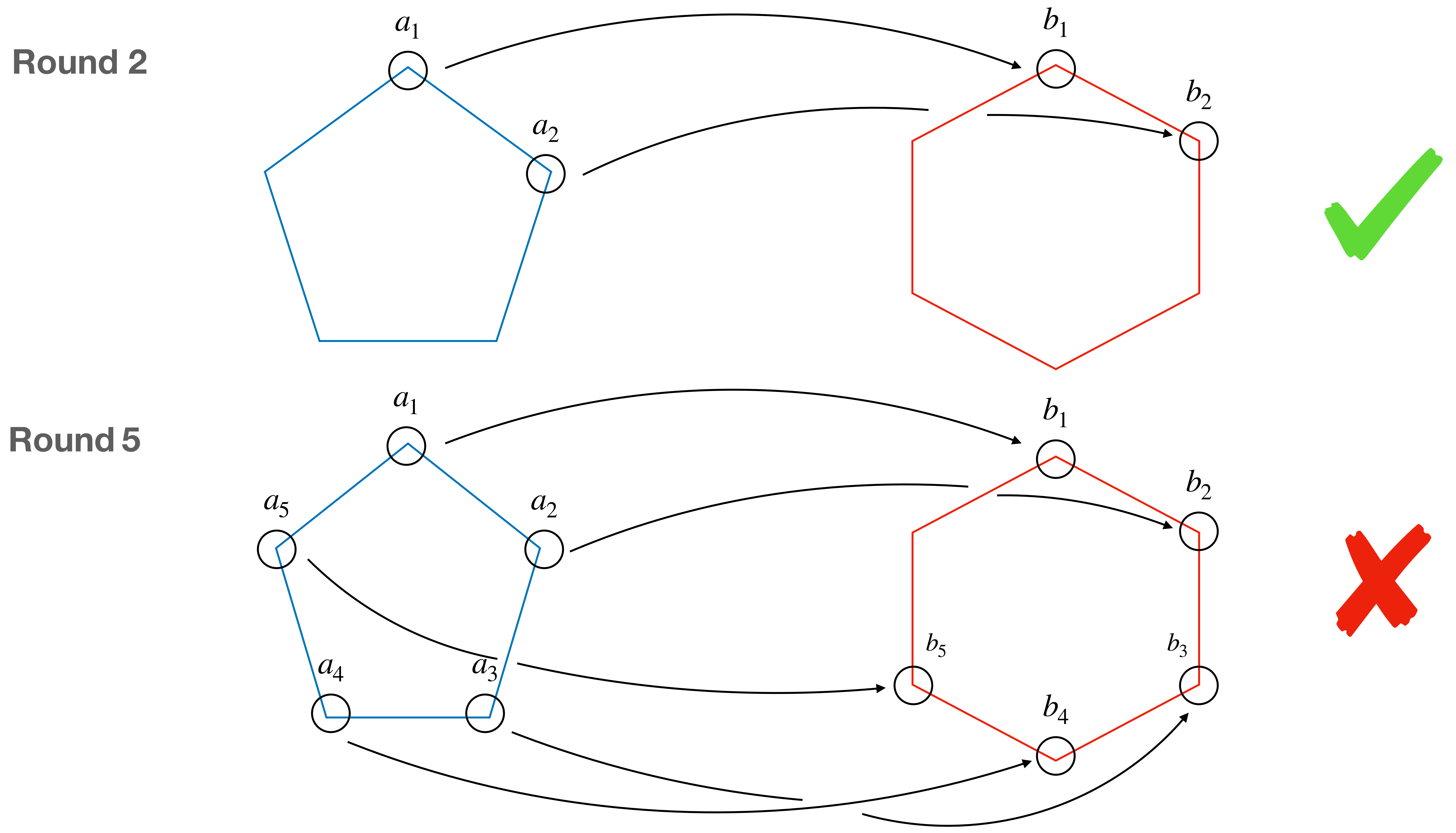
Spoiler chooses a_1

Duplicator responds b_1

Round 2



Ehrenfeucht-Fraïssé Game between ⬠ and ⬡



Duplicator winning implies that $\mathscr{A}$ and $\mathscr{B}$ are related in $\mathscr{L}$

Harder game for Duplicator means more expressive $\mathscr{L}$

Reference	Game	Corresponding Logical Relation
Fraïssé 1950’s	$\exists\mathsf{EF}_k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \Rrightarrow_{\exists^+\mathscr{L}_k} \mathscr{B}$

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Fraïssé 1950's	$(\exists)\mathrm{EF}_k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \Rrightarrow_{\exists^+\mathscr{L}_k} \mathscr{B} / \mathscr{A} \equiv_{\mathscr{L}_k} \mathscr{B}$
Kolaitis & Vardi 1992	$\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \Rrightarrow_{\exists^+\mathscr{L}^k} \mathscr{B}$
Hella 1996	$\mathrm{Bij}_k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \equiv_{\mathscr{C}^k} \mathscr{B}$
Hella 1996	$\mathrm{Bij}_n^k(\mathscr{A}, \mathscr{B})$	$\mathscr{A} \equiv_{\mathscr{L}^k(\mathscr{Q}_n)} \mathscr{B}$

← $\exists^{\geq k}$

← $Q_{\mathscr{K}}$

The Rise of Game Comonads

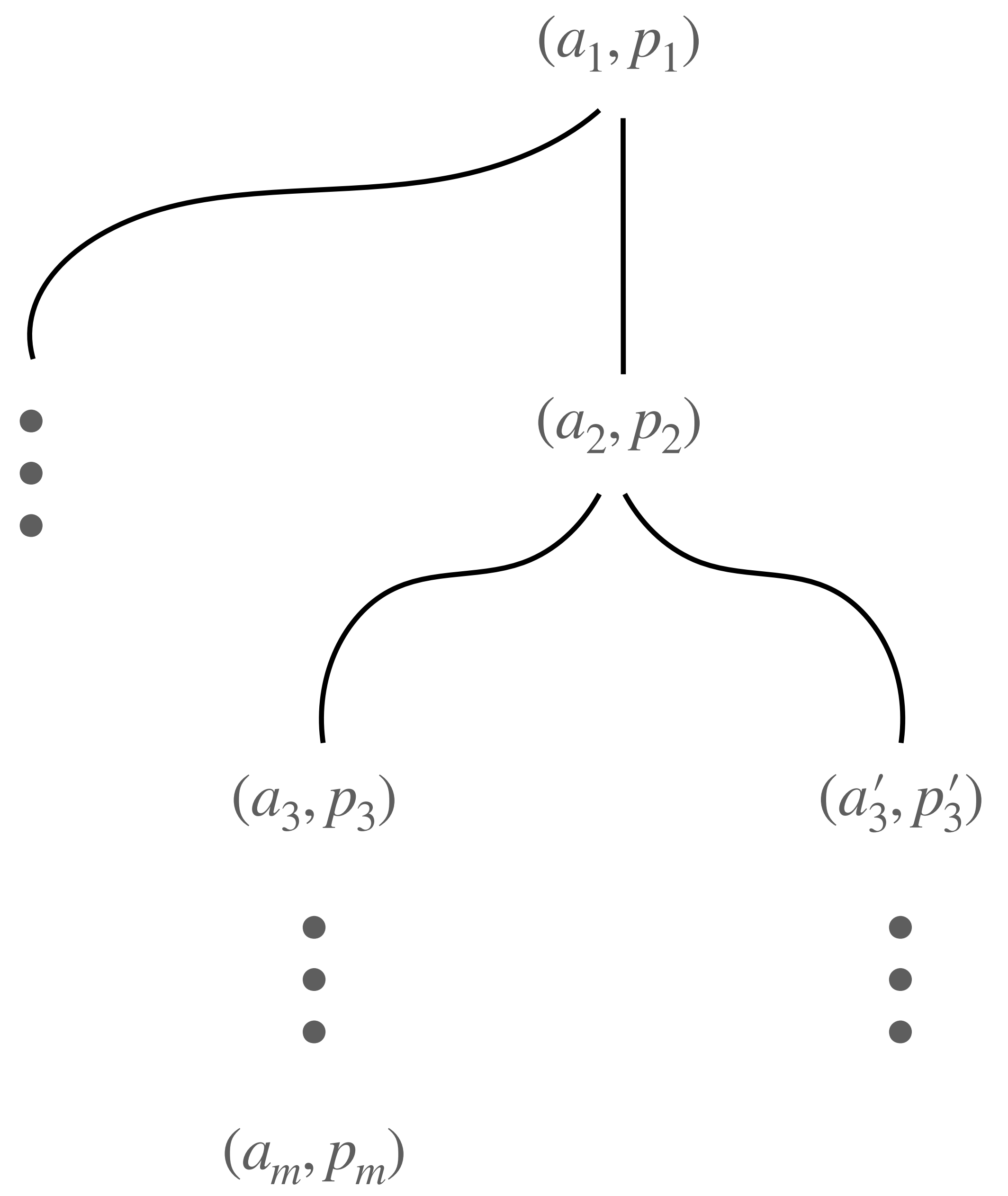
Can we connect these two categorically?

$$(\mathscr{R}(\sigma), \to\, , \cong\,) \rightleftarrows (\mathscr{R}(\sigma), \Rrightarrow_{\mathscr{L}}\, , \equiv_{\mathscr{L}}\,)$$

Abramsky, Dawar & Wang's Pebbling Comonad

$\mathbb{P}_k\mathscr{A}$ = $\langle (A \times [k])^+$, relations from $\mathscr{A}$ according to tree structure$\rangle$

$(A \times [k])^+$ is the universe of *histories* of Spoiler moves in the k-pebble game



Relations on $\mathbb{P}_k\mathscr{A}$ are controlled by the last element and the tree structure

Given some a history of moves $s = [(a,2),(b,3),(c,1),(d,2),(e,1)] \in \mathbb{P}_k\mathscr{A}$

Last pebbled element is extracted by the function $\epsilon(s) = e$

Elements relevant for relations are the "live" ones $[(a,2),(b,3),(c,1),(d,2),(e,1)]$

Live prefixes of s are $[(a,2),(b,3)]$, $[(a,2),(b,3),(c,1),(d,2)]$ and s

$(s_1, \ldots s_m) \in R^{\mathbb{P}_k\mathscr{A}}$ iff $(\epsilon(s_1), \ldots \epsilon(s_m)) \in R^{\mathscr{A}}$ and
$\forall i, j$ s_i and s_j are related in the live prefix relation.

Abramsky, Dawar & Wang's Pebbling Comonad

$\mathbb{P}_k\mathscr{A} = \langle (A \times [k])^+$, relations from $\mathscr{A}$ according to tree structure$\rangle$

Counit $\epsilon : \mathbb{P}_k\mathscr{A} \to \mathscr{A}$

$$\epsilon([(a_1, p_1), \ldots, (a_m, p_m)]) = a_m$$

Comultiplication $\delta : \mathbb{P}_k\mathscr{A} \to \mathbb{P}_k\mathbb{P}_k\mathscr{A}$

$$\delta([(a_1, p_1), \ldots, (a_m, p_m)]) = [(s_1, p_1), \ldots(s_m, p_m)]$$

where $s_i = [(a_1, p_1)\ldots, (a_i, p_i)]$

Abramsky, Dawar & Wang's Pebbling Comonad

$\mathbb{P}_k\mathscr{A} = \langle (A \times [k])^+$, relations from $\mathscr{A}$ according to tree structure$\rangle$

Kleisli Category $\mathscr{K}(\mathbb{P}_k)$

$\mathbb{P}_k\mathscr{A} \to \mathscr{B} \iff$ Duplicator has a winning strategy for $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$

$\mathscr{A} \cong_{\mathscr{K}(\mathbb{P}_k)} \mathscr{B} \iff$ Duplicator has a winning strategy for $\text{Bij}_k(\mathscr{A}, \mathscr{B})$

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Coalgebras

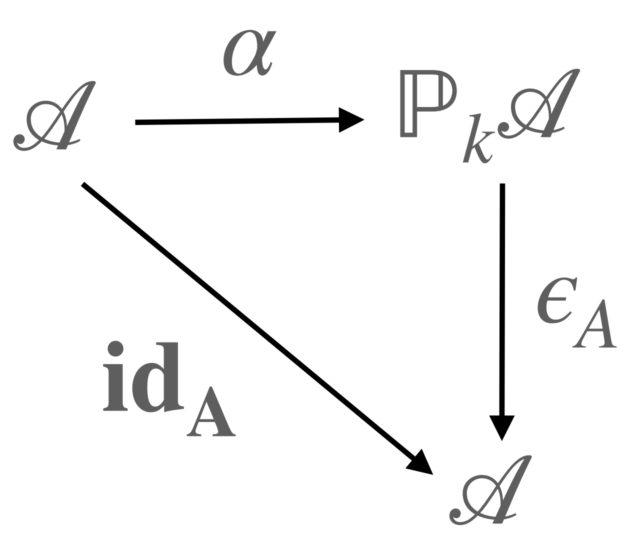
$\alpha : \mathscr{A} \to \mathbb{P}_k\mathscr{A} \iff \mathscr{A}$ has a tree decomposition of width k

A surprising discovery: coalgebras are decompositions

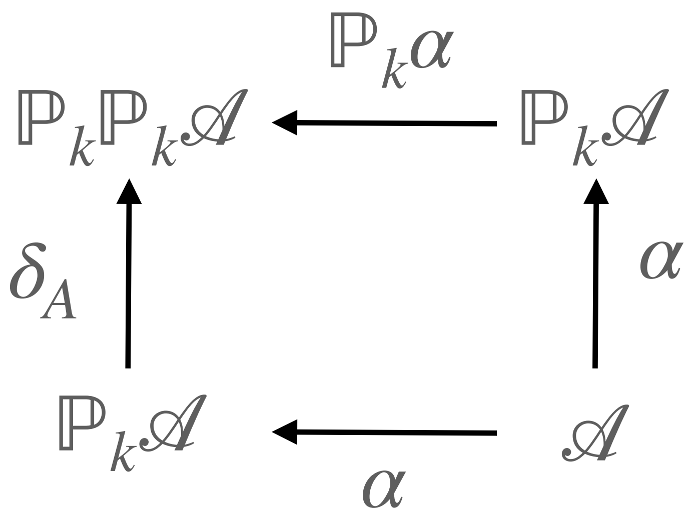
Coalgebras of a comonad

Morphisms $\alpha : \mathscr{A} \to \mathbb{P}_k\mathscr{A}$ satisfying two laws

Counit Law:

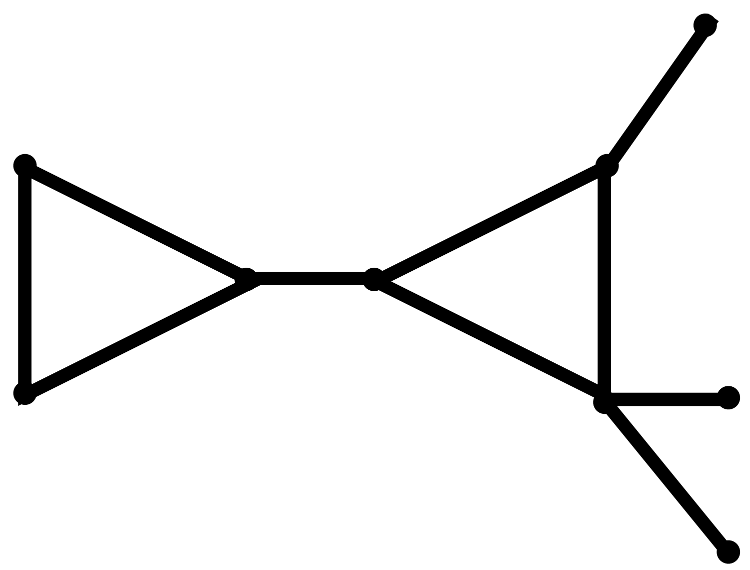


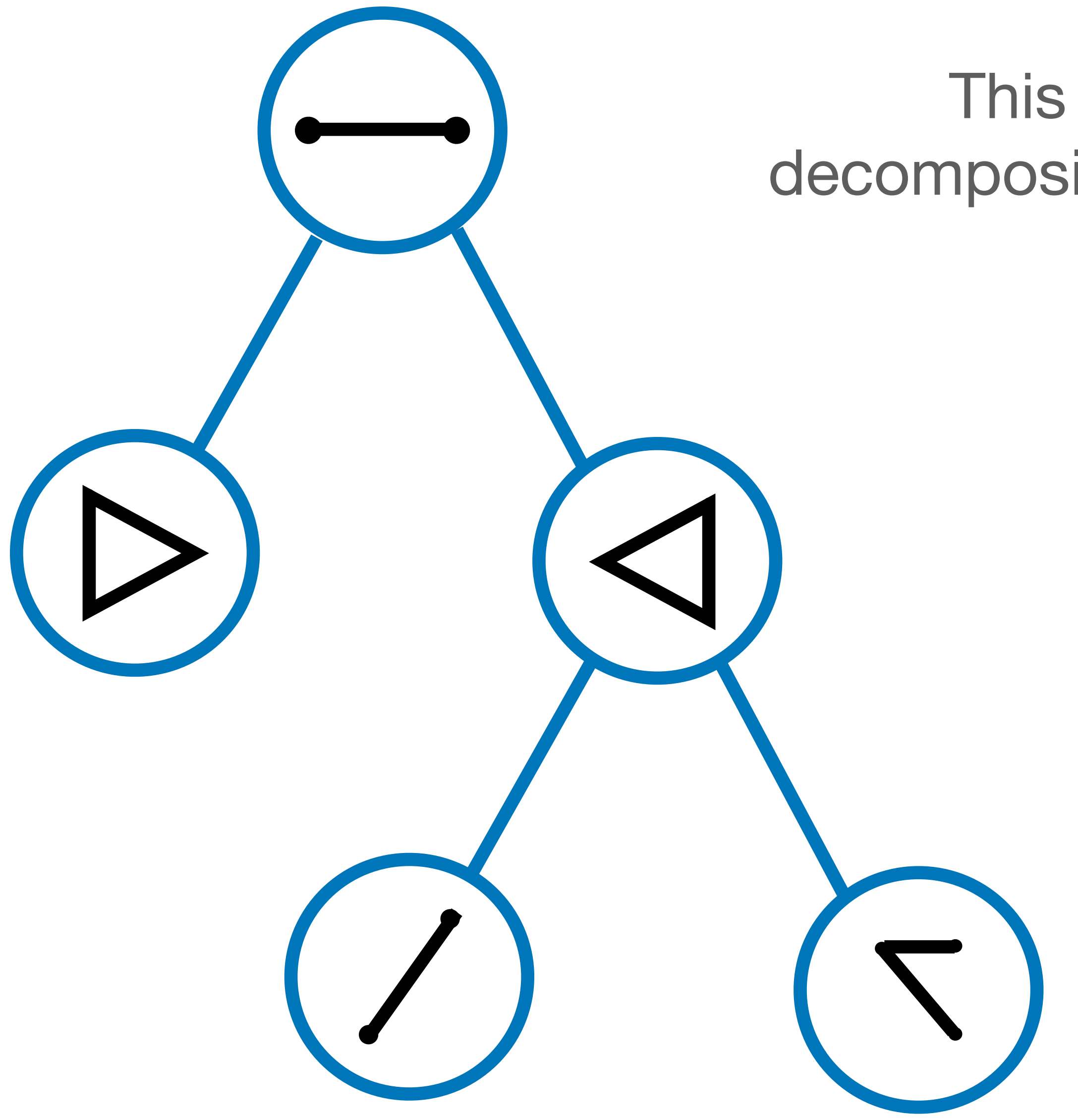
Comultiplication Law:



Tree decompositions of a relational structure

Robertson & Seymour pioneered the study of taking a relational structure and studying its decompositions such as that below





This is a tree decomposition of width 2

Abramsky, Dawar & Wang 2017

There exists $\alpha : \mathscr{A} \to \mathbb{P}_k\mathscr{A}$ a coalgebra

$\iff$ $\mathscr{A}$ has a tree decomposition of width k

Can we connect these two categorically? **Yes!**

$$(\mathscr{R}(\sigma), \rightarrow, \cong) \rightleftarrows (\mathscr{R}(\sigma), \Rrightarrow_{\mathscr{L}}, \equiv_{\mathscr{L}})$$

Can we connect these two categorically? **Yes!**

$$(\mathscr{R}(\sigma), \rightarrow, \cong) \underset{\longleftarrow}{\overset{\longrightarrow}{\mathbb{P}_k}} (\mathscr{R}(\sigma), \Rrightarrow_{\exists^+\mathscr{L}^k}, \equiv_{\mathscr{C}^k})$$

Where $\mathbb{P}_k$ is graded in k which controls the number of variables in the underlying logic

Reference	Comonad	Related games	Logical Resource	Coalgebra parameter
ADW 2017	$\mathbb{P}_k$	Pebble games	Variables	Treewidth

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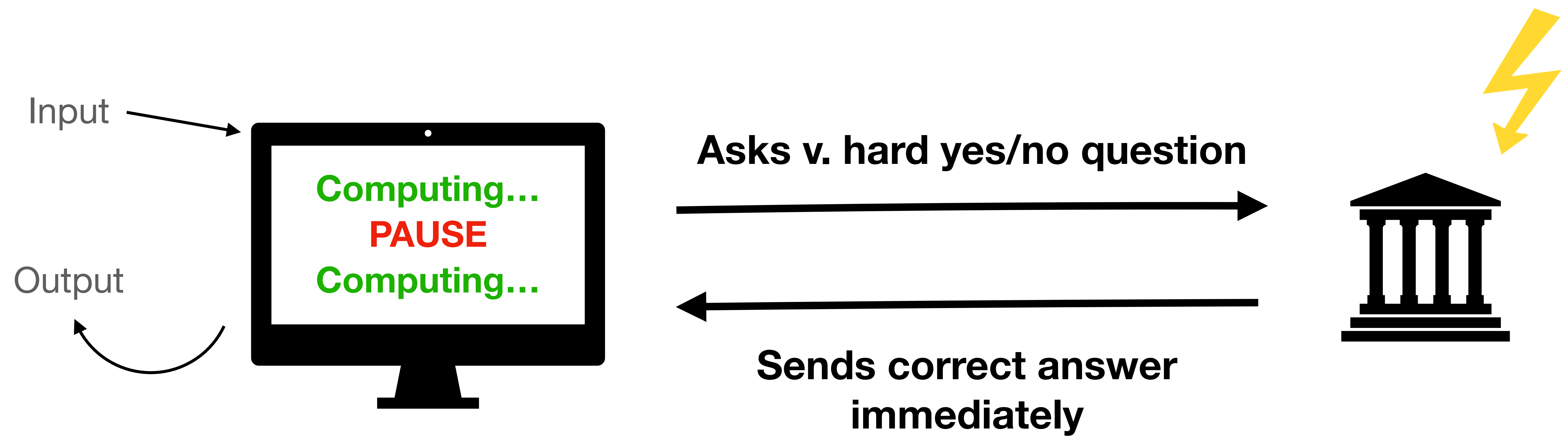
ADW 2017	$\mathbb{P}_k$	Pebble games	Variables	Treewidth
Abramsky & Shah 2018	$\mathbb{E}_n$	Ehrenfeucht-Fraïssé	Quantifier depth	Treedepth
Abramsky & Shah 2018	$\mathbb{M}_n$	Modal bisimulation	Modal depth	Modal unfolding depth

$\rightarrow_{\mathscr{K}}$ is $\Rrightarrow_{\exists^{+}\mathscr{L}}$
and
$\cong_{\mathscr{K}}$ is $\Rrightarrow_{\mathscr{L}(\exists^{\geq m})}$

My work on game comonads and quantifiers

Need more power? Consult an oracle!



Oracle computation exists everywhere in computer science, cryptography and complexity theory (and Ancient Greece!)

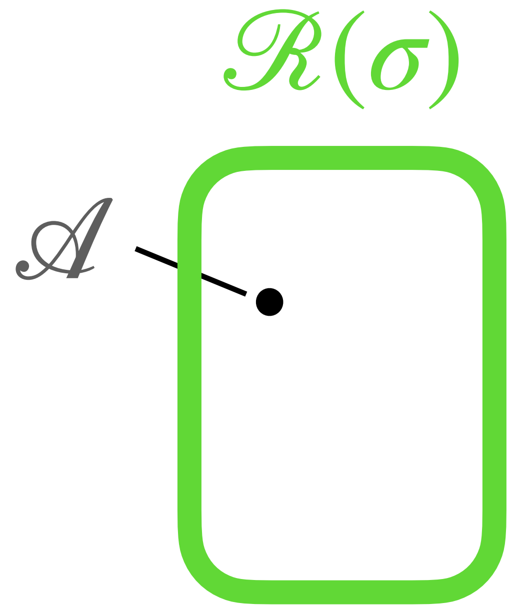
In the world of logic, oracles are added using “generalised quantifiers” (due to Per Lindstrom)

Some work had already been done (by Hella) giving a two-way game for logics extended by these oracles.

$$\text{Duplicator wins } \mathrm{Bij}_n^k(\mathscr{A}, \mathscr{B}) \iff \mathscr{A} \equiv_{\mathscr{L}^k(\mathbf{Q}_n)} \mathscr{B}$$

Quantifiers as a Resource

Building a new quantifier



A relational structure $\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R \in \sigma} \rangle \in \mathscr{R}(\sigma)$

Building a new quantifier

$$\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R \in \sigma} \rangle \in \mathscr{R}(\sigma)$$

A class of structures $\mathscr{K} \subset \mathscr{R}(\tau)$

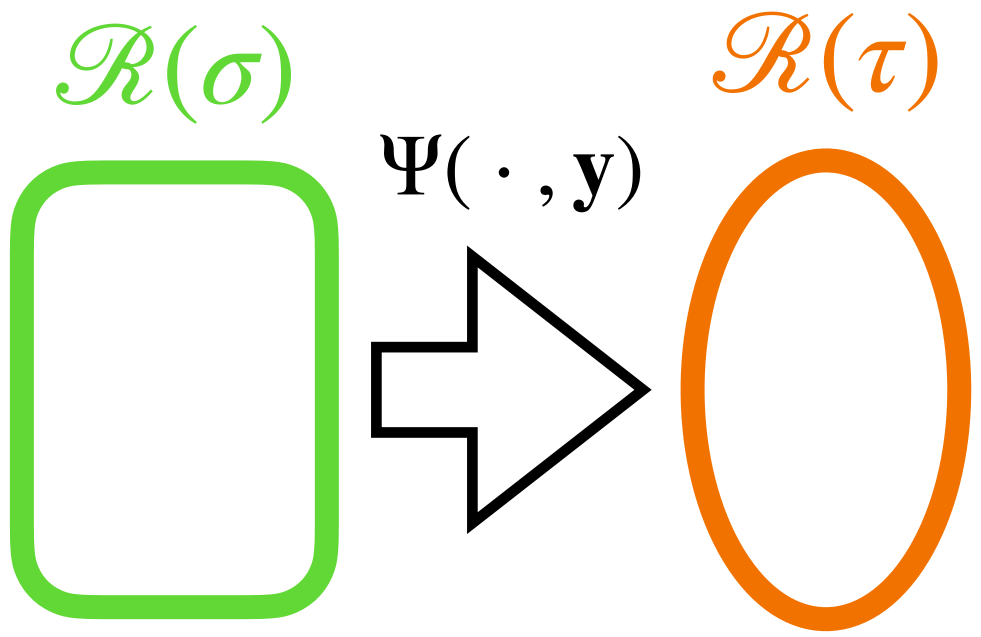
Building a new quantifier

$$\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R \in \sigma} \rangle \in \mathscr{R}(\sigma)$$

$$\mathscr{K} \subset \mathscr{R}(\tau)$$

An interpretation

$$\Psi(\mathbf{x}, \mathbf{y}) = \langle \psi_T(\mathbf{x}_T, \mathbf{y}_T) \rangle_{T \in \tau}$$



Building a new quantifier

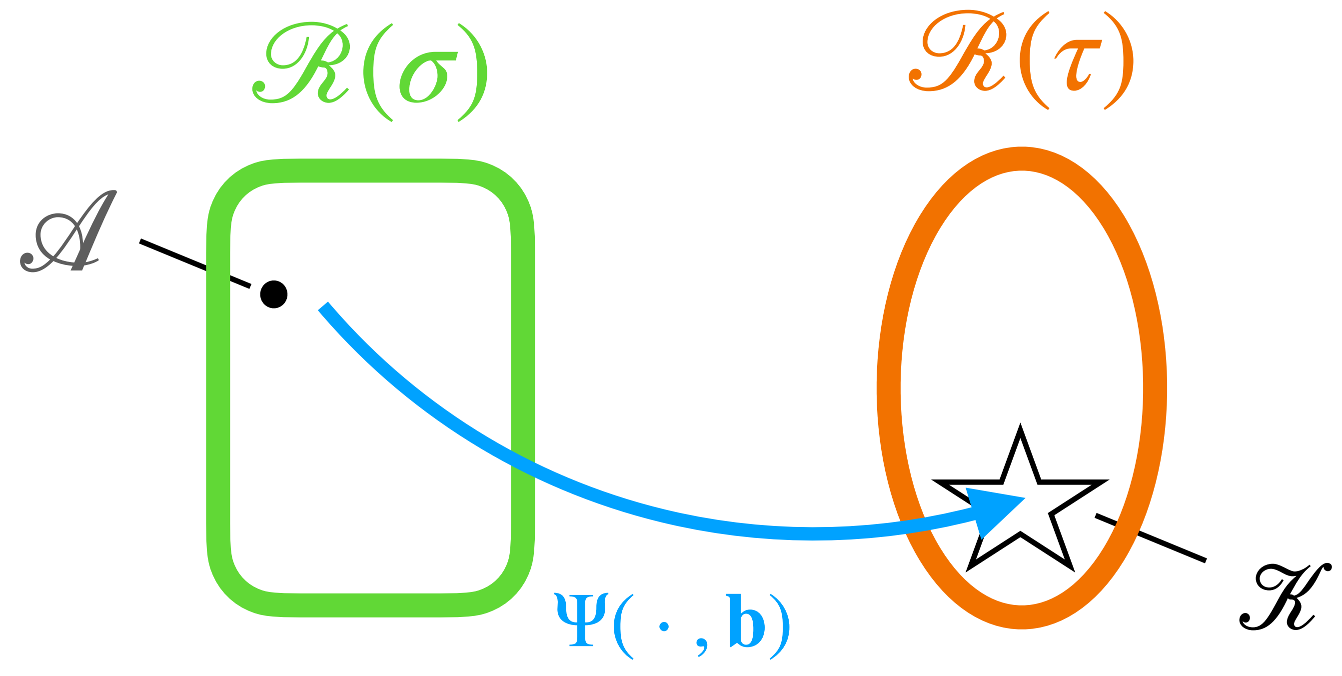
$$\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R\in\sigma}\rangle \in \mathscr{R}(\sigma)$$

$$\mathscr{K} \subset \mathscr{R}(\tau)$$

$$\Psi(\mathbf{x}, \mathbf{y}) = \langle \psi_T(\mathbf{x}_T, \mathbf{y}_T)\rangle_{T\in\tau}$$

A new quantifier

$$\mathscr{A}, \mathbf{b} \vDash Q_{\mathscr{K}}\mathbf{x} \,.\, \Psi(\mathbf{x}, \mathbf{y})$$



Building a new quantifier

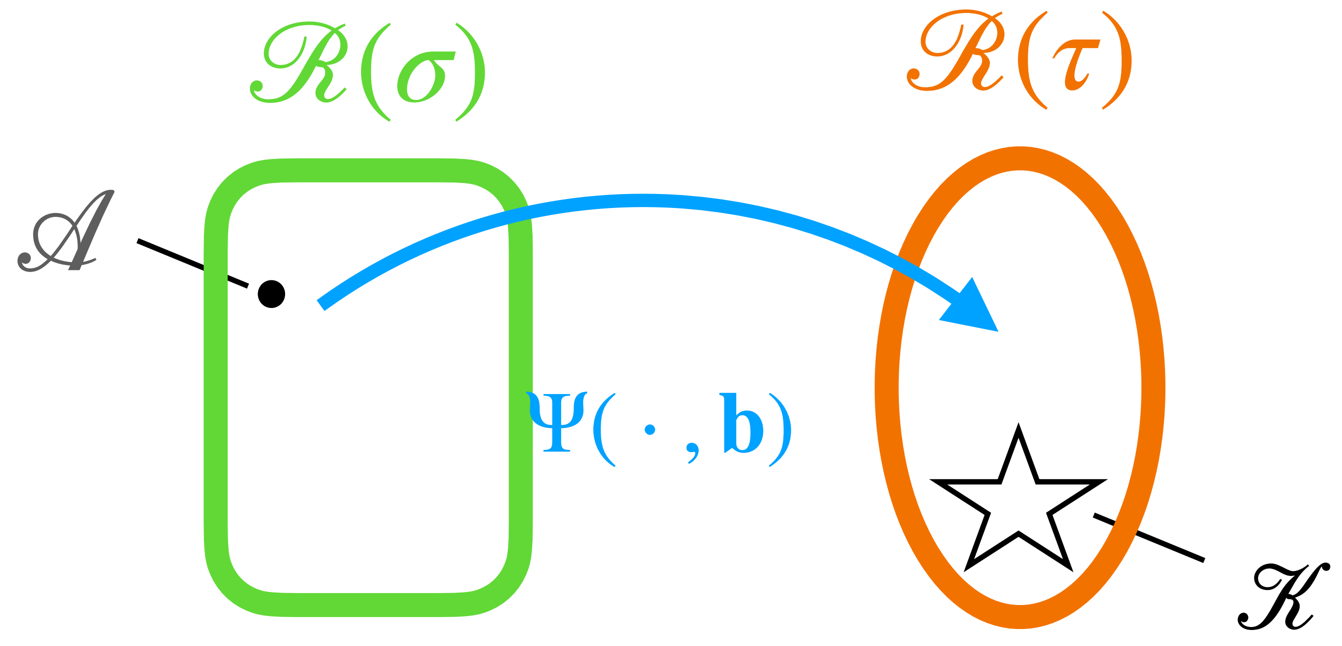
$$\mathscr{A} = \langle A, (R^{\mathscr{A}})_{R \in \sigma} \rangle \in \mathscr{R}(\sigma)$$

$$\mathscr{K} \subset \mathscr{R}(\tau)$$

$$\Psi(\mathbf{x}, \mathbf{y}) = \langle \psi_T(\mathbf{x}_T, \mathbf{y}_T) \rangle_{T \in \tau}$$

A new quantifier

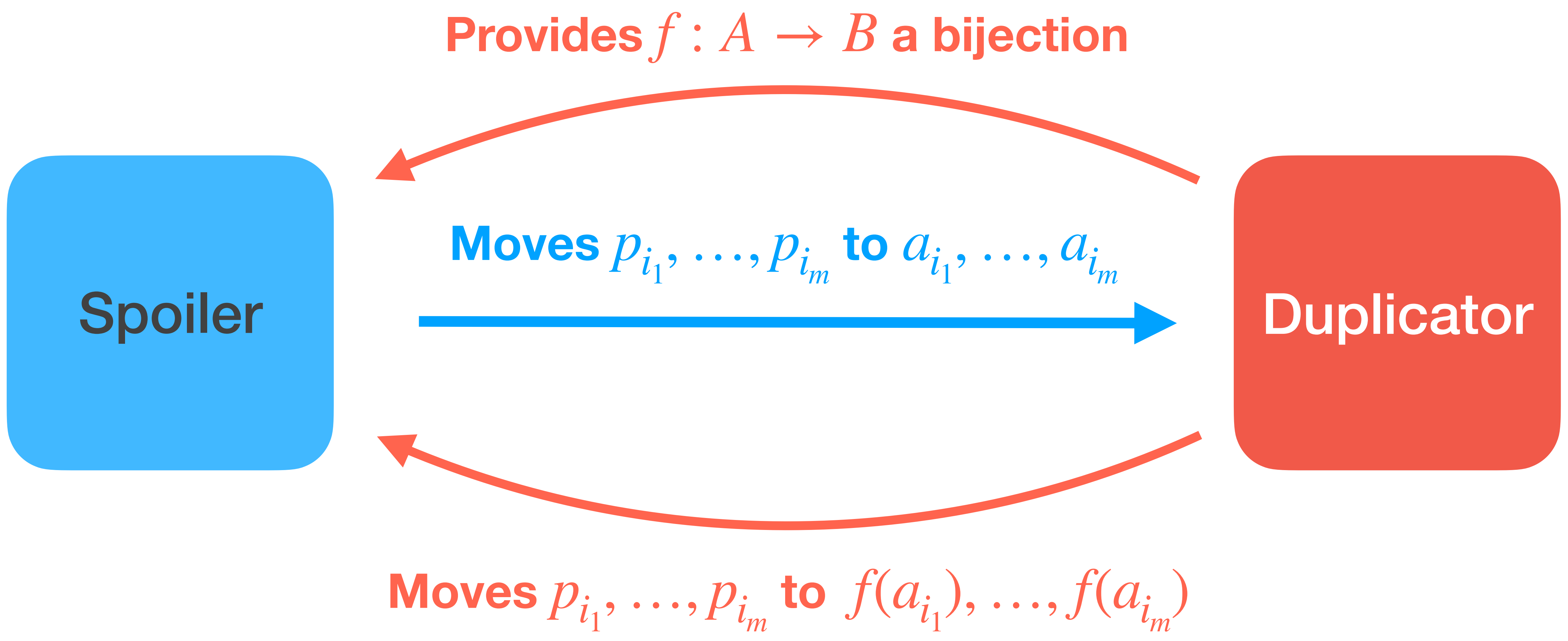
$$\mathscr{A}, \mathbf{b} \not\models Q_{\mathscr{K}} \mathbf{x} \,.\, \Psi(\mathbf{x}, \mathbf{y})$$



A game to control these new quantifiers

$\mathscr{L}^k(\mathbf{Q}_n)$ is k-variable infinitary first-order logic extended by quantifiers of isomorphism-closed classes of structures with no relation of arity $> n$

$\mathrm{Bij}_n^k(\mathscr{A}, \mathscr{B})$ game (Hella 1996)

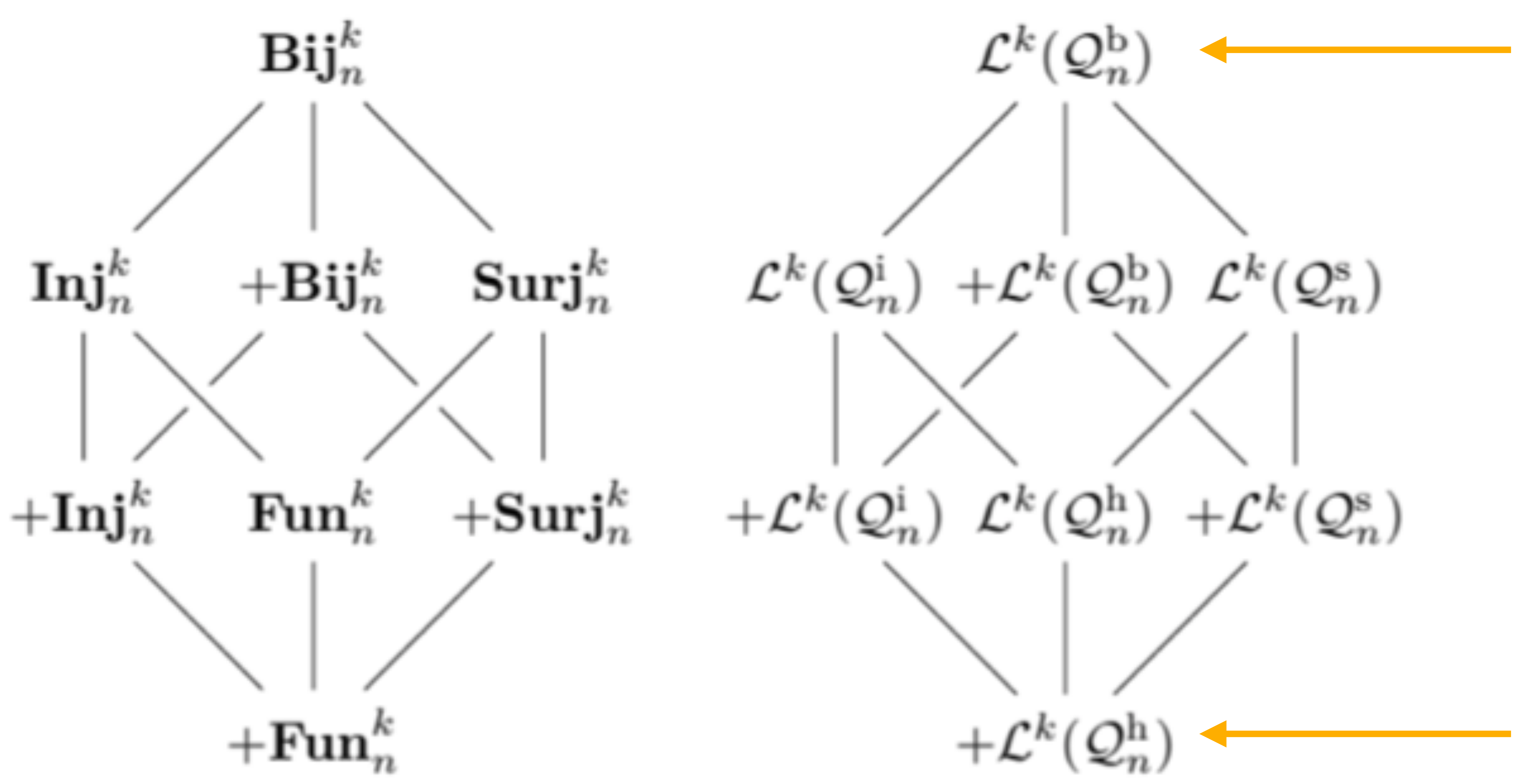


Theorem (Hella 1996)

Duplicator has a winning strategy for $\mathrm{Bij}_n^k(\mathscr{A}, \mathscr{B})$ if and only if $\mathscr{A} \equiv_{\mathscr{L}^k(\mathbf{Q}_n)} \mathscr{B}$

$\mathbb{G}_{n,k}$: a comonad for quantifiers

Improving our understanding of these oracles



All “n-ary” quantifiers (including $\sharp$ for n = 1)

“n-ary” hom-closed quantifiers (including $\exists$ for n = 1)

Theorem 15 (Ó C. & Dawar, 2021)

For a game $\mathscr{G}$ from the left-hand diagram, Duplicator wins $\mathscr{G}(\mathscr{A}, \mathscr{B})$ if and only if $\mathscr{A} \Rrightarrow_{\mathscr{L}^{\mathscr{G}}} \mathscr{B}$ where $\mathscr{L}^{\mathscr{G}}$ is the corresponding logic from the right-hand diagram

Constructing a new comonad from an old one

Pebbling Comonad

$\mathbb{P}_k\mathscr{A} \to \mathscr{B} \iff \exists\mathsf{Peb}_k(\mathscr{A},\mathscr{B}) \iff \mathscr{A} \Rrightarrow_{\exists^+\mathscr{L}^k} \mathscr{B}$

$\mathbb{P}_k\mathscr{A} \cong \mathbb{P}_k\mathscr{B} \iff \mathsf{Bij}_k(\mathscr{A},\mathscr{B}) \iff \mathscr{A} \equiv_{\mathscr{L}^k(\sharp)} \mathscr{B}$

New Generalised Quantifier Comonad

$\mathbb{G}_{n,k}\mathscr{A} := \mathbb{P}_k\mathscr{A}/\approx_n$

$+\mathsf{Fun}_n^k(\mathscr{A},\mathscr{B}) \iff \mathscr{A} \Rrightarrow_{+\mathscr{L}^k(\mathbf{Q}_n^h)} \mathscr{B} \iff \mathbb{G}_{n,k}\mathscr{A} \to \mathscr{B}$

$\mathsf{Bij}_n^k(\mathscr{A},\mathscr{B}) \iff \mathscr{A} \equiv_{\mathscr{L}^k(\mathbf{Q}_n)} \mathscr{B} \iff \mathbb{G}_{n,k}\mathscr{A} \cong \mathbb{G}_{n,k}\mathscr{B}$

Lemma 20 (Ó C. & Dawar, 2021)

Duplicator has a winning strategy for $+\mathsf{Fun}_n^k(\mathscr{A},\mathscr{B})$ if and only if she has an "n-consistent" winning strategy for $\exists\mathsf{Peb}_k(\mathscr{A},\mathscr{B})$

Then defined $\approx_n$ a relation on any $\mathbb{P}_k\mathscr{A}$ such that

$\mathbb{P}_k\mathscr{A}/\approx_n \to \mathscr{B} \iff$ Duplicator wins $\exists\mathsf{Peb}_k(\mathscr{A},\mathscr{B})$ n-consistently

Consequences of this new comonad

$$\mathbb{G}_{n,k}\mathscr{A} = \mathbb{P}_k\mathscr{A} / \approx_n$$

Kleisli Category $\mathscr{K}(\mathbb{G}_{n,k})$

$\mathbb{G}_{n,k}\mathscr{A} \to \mathscr{B} \iff$ Duplicator has a winning strategy for $+\mathrm{Fun}_n^k(\mathscr{A}, \mathscr{B})$

$\mathscr{A} \cong_{\mathscr{K}(\mathbb{G}_{n,k})} \mathscr{B} \iff$ Duplicator has a winning strategy for $\mathrm{Bij}_n^k(\mathscr{A}, \mathscr{B})$

Coalgebras

$\alpha : \mathscr{A} \to \mathbb{G}_{n,k}\mathscr{A} \iff \mathscr{A}$ has an extended tree decomposition of width k and arity n

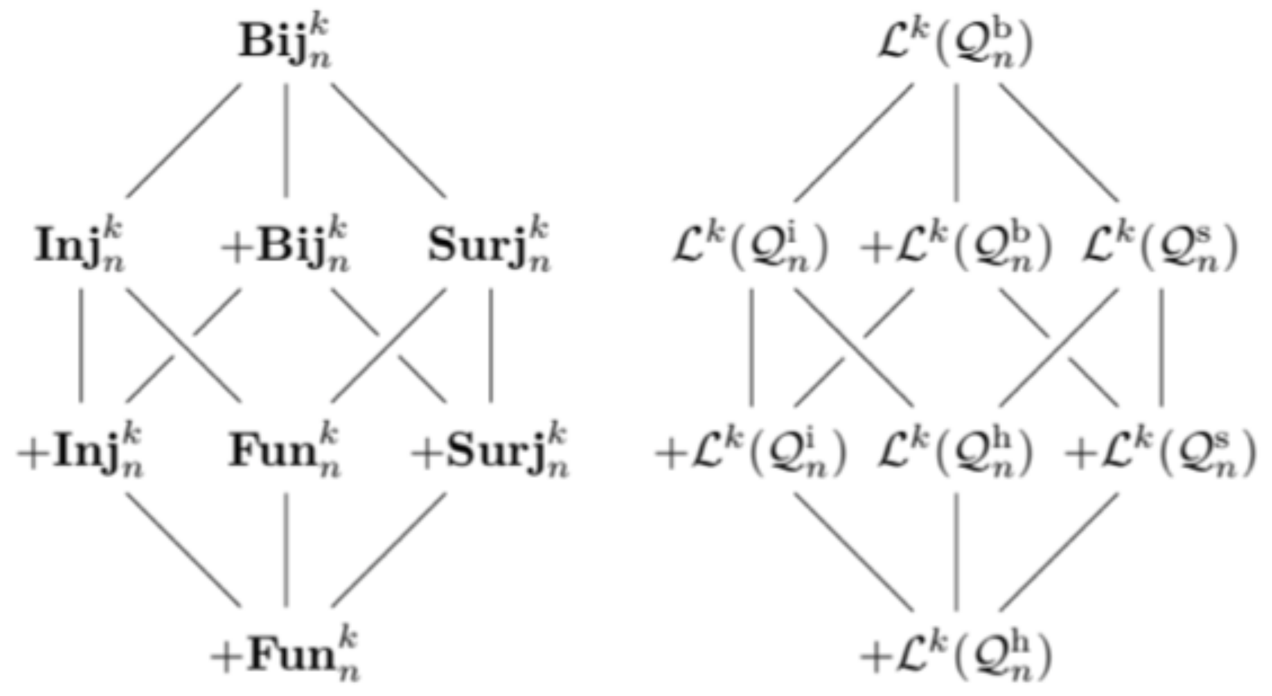
Conclusions & Future Directions

A much clearer understanding of the relation between quantifiers and the Kleisli Category of game comonads

$\rightarrow_{\mathscr{K}}$ is $\Rrightarrow_{\exists^{+}\mathscr{L}}$

and

$\cong_{\mathscr{K}}$ is $\Rrightarrow_{\mathscr{L}(\exists^{\geq m})}$



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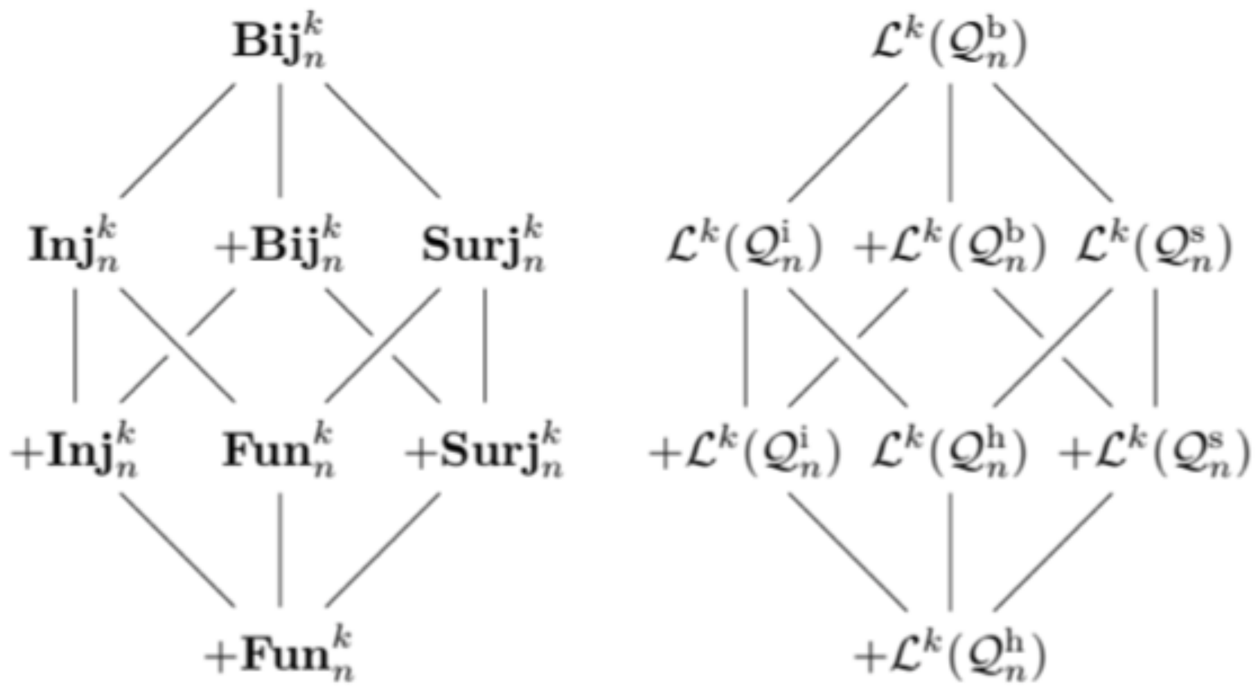


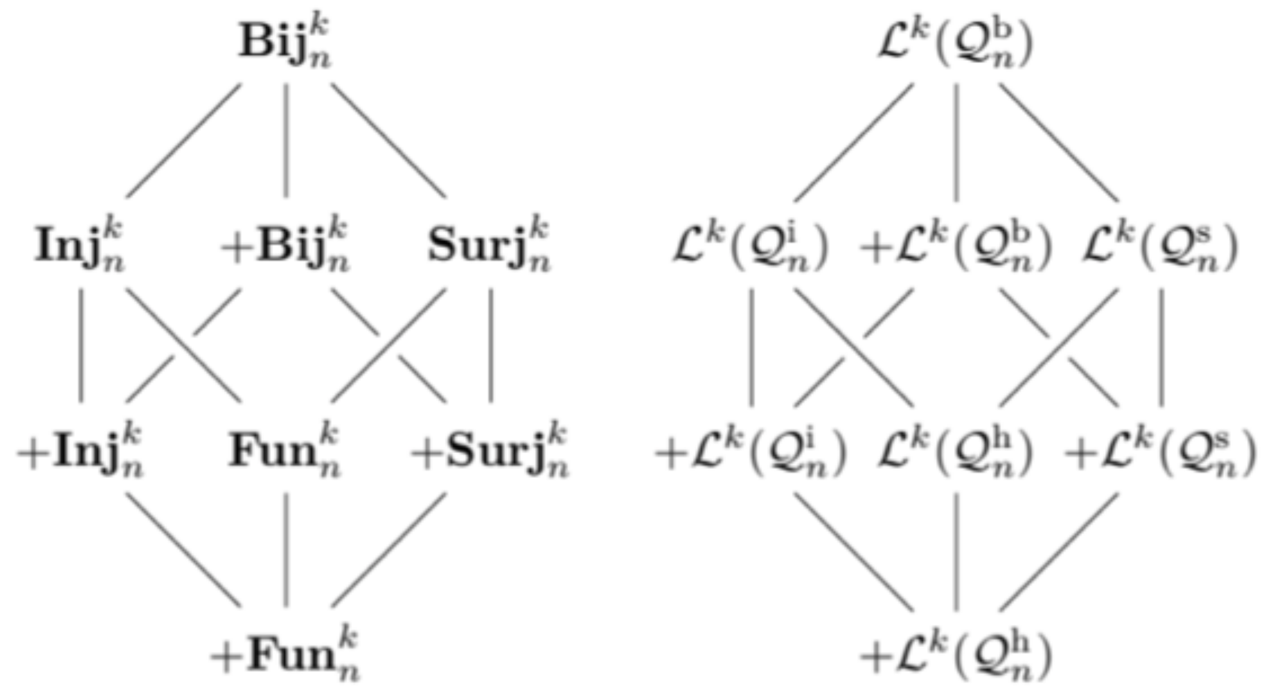
A method for constructing new games and new game comonads from old ones.
Can we turn more game theoretic translations into category theory?

$$\mathbb{G}_{n,k}\mathscr{A} = \mathbb{P}_k\mathscr{A} / \approx_n$$

Conclusions & Future Directions

A much clearer understanding of the relation between quantifiers and the Kleisli Category of game comonads

$\to_{\mathscr{K}}$ is $\Rrightarrow_{\exists^{+}\mathscr{L}}$
and
$\cong_{\mathscr{K}}$ is $\Rrightarrow_{\mathscr{L}(\exists^{\geq m})}$



A method for constructing new games and new game comonads from old ones.
Can we turn more game theoretic translations into category theory?

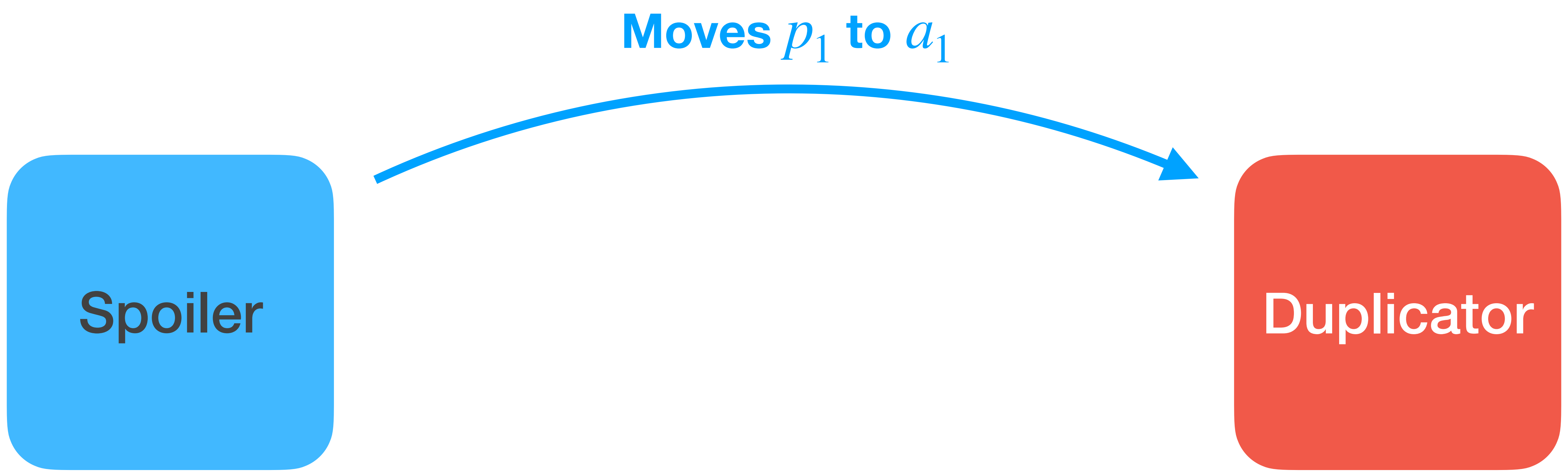
$$\mathbb{G}_{n,k}\mathscr{A} = \mathbb{P}_k\mathscr{A}/\approx_n$$

Some of the candidate logics for P (e.g. rank logic) are defined using classes of generalised quantifiers.
Can techniques from this work help us to make new comonads for these logics?

Extra material if there’s time

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$

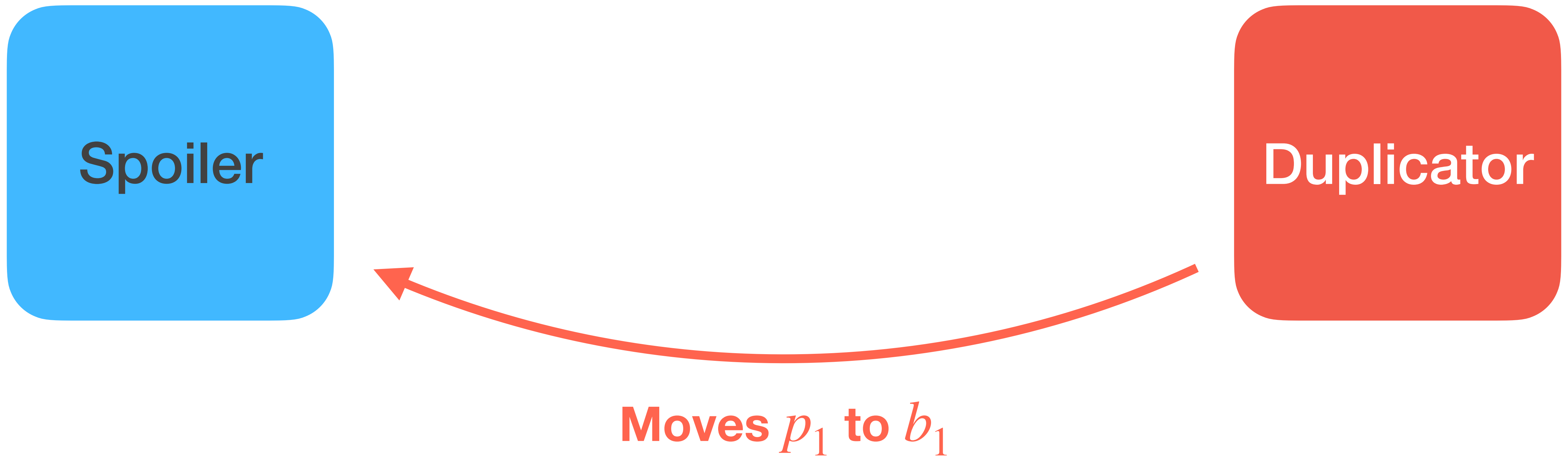


A homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$$[(p_1, a_1)] \quad \mapsto$$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$

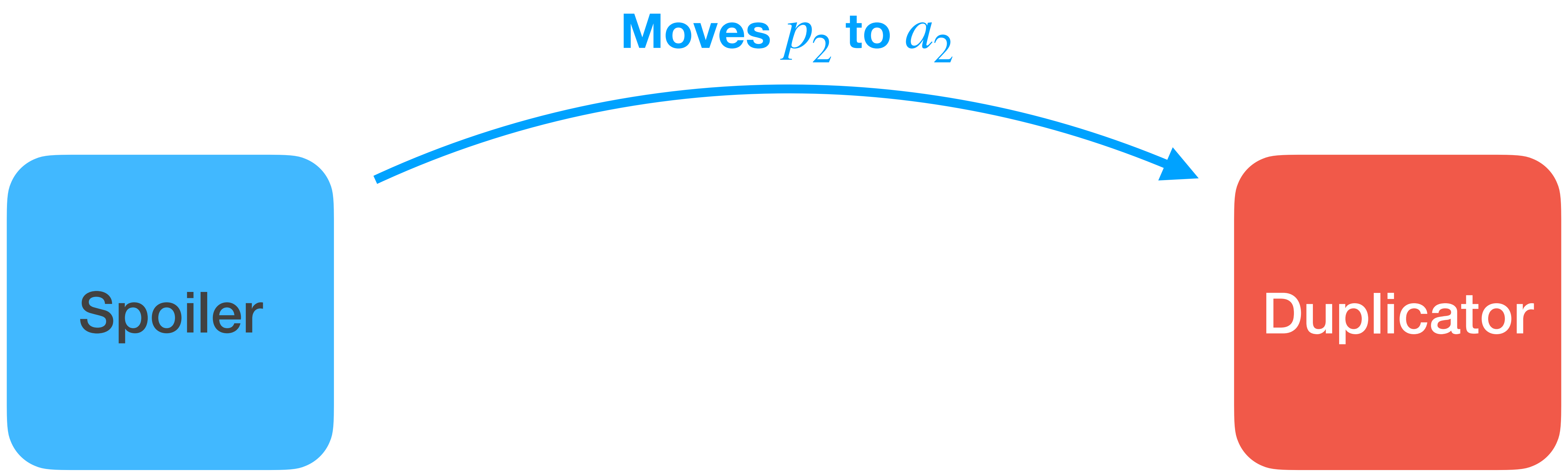


A homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$$[(p_1, a_1)] \quad \mapsto \quad b_1$$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$



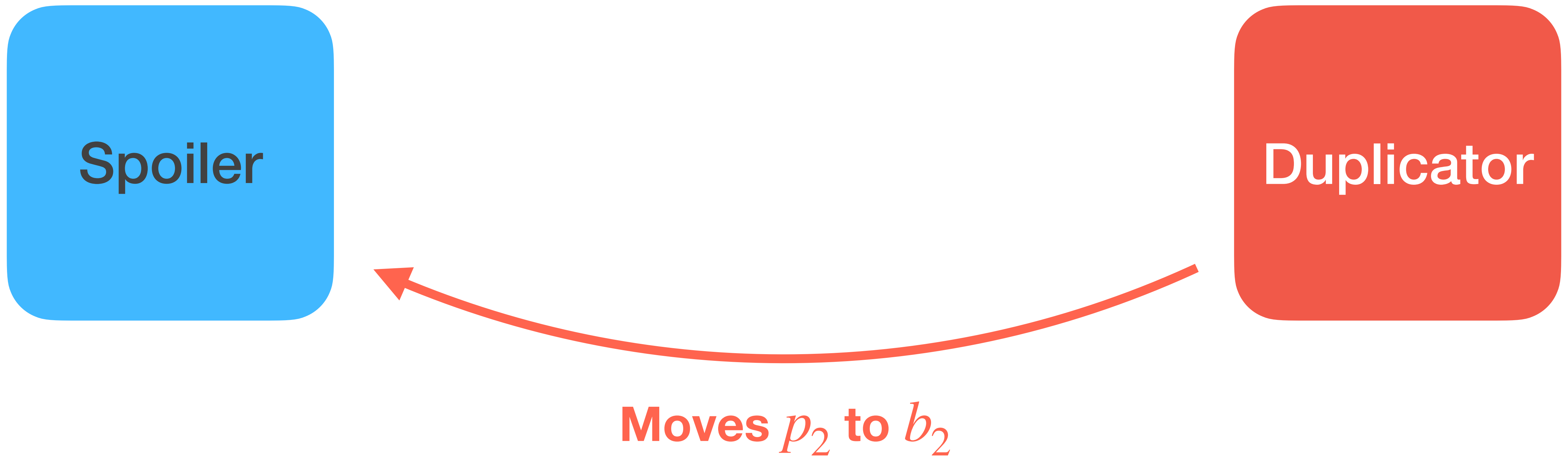
A homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$$[(p_1, a_1)] \quad \mapsto \quad b_1$$

$$[(p_1, a_1), (p_2, a_2)] \quad \mapsto$$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$



A homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

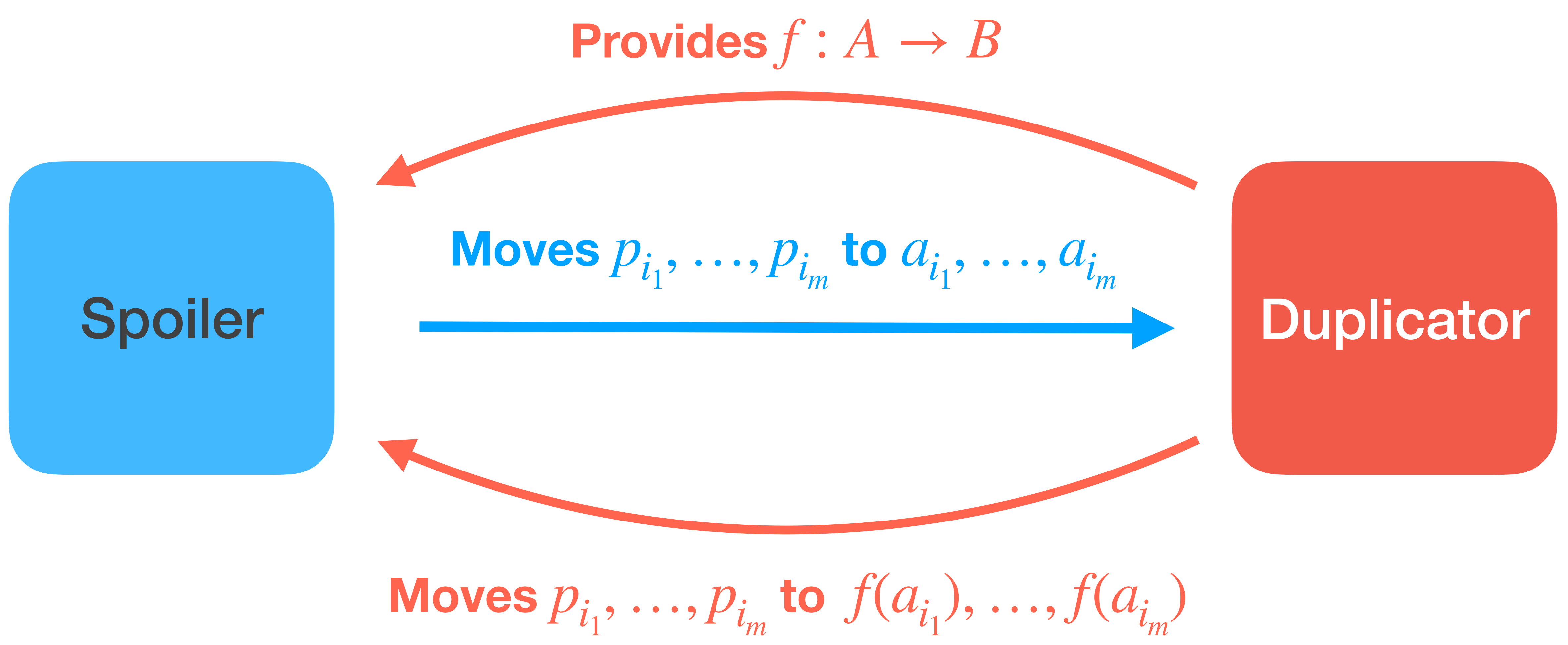
$$[(p_1, a_1)] \mapsto b_1$$

$$[(p_1, a_1), (p_2, a_2)] \mapsto b_2$$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's strategy in $+\mathrm{Fun}_n^k(\mathscr{A}, \mathscr{B})$

A homomorphism $\mathbb{G}_{n,k}\mathscr{A} \to \mathscr{B}$



???

Lemma 20 (Ó C. & Dawar, 2021)

Duplicator has a winning strategy for $+\mathrm{Fun}_n^k(\mathscr{A}, \mathscr{B})$ if and only if she has an "n-consistent" winning strategy for $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$

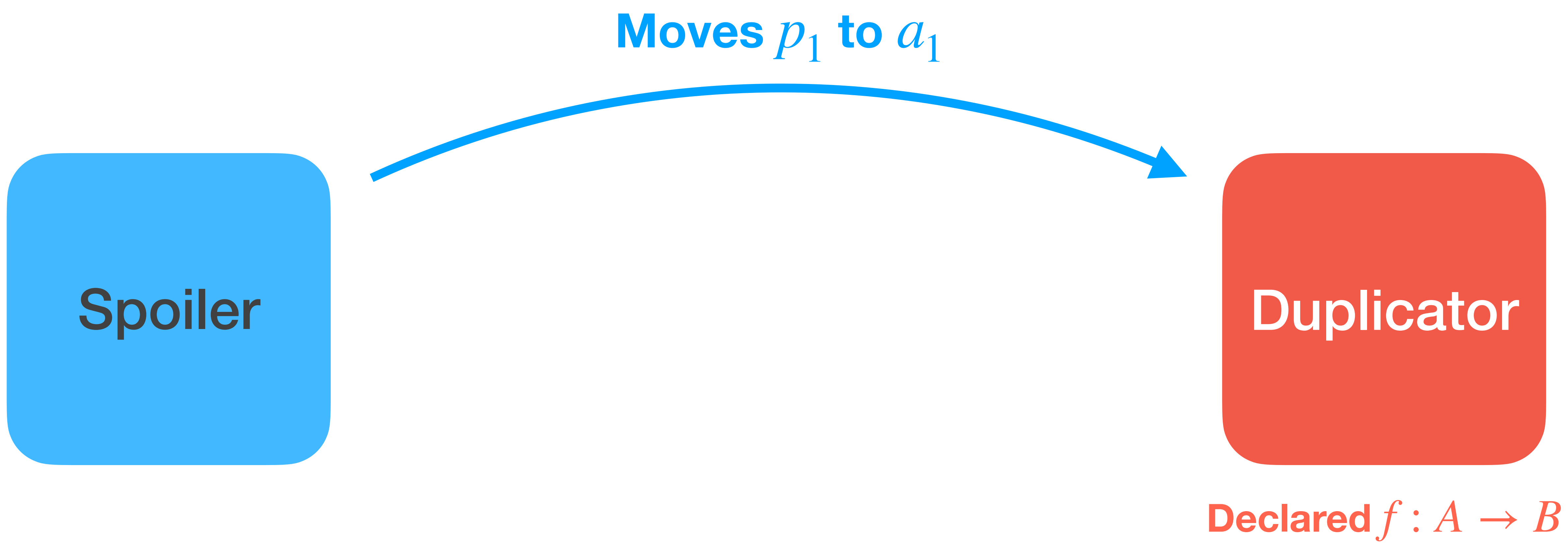
Creating a new comonad from $\mathbb{P}_k$

Duplicator's "n-consistent" strategy for $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$

A "special" homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's "n-consistent" strategy for $\exists\mathrm{Peb}_k(\mathscr{A}, \mathscr{B})$



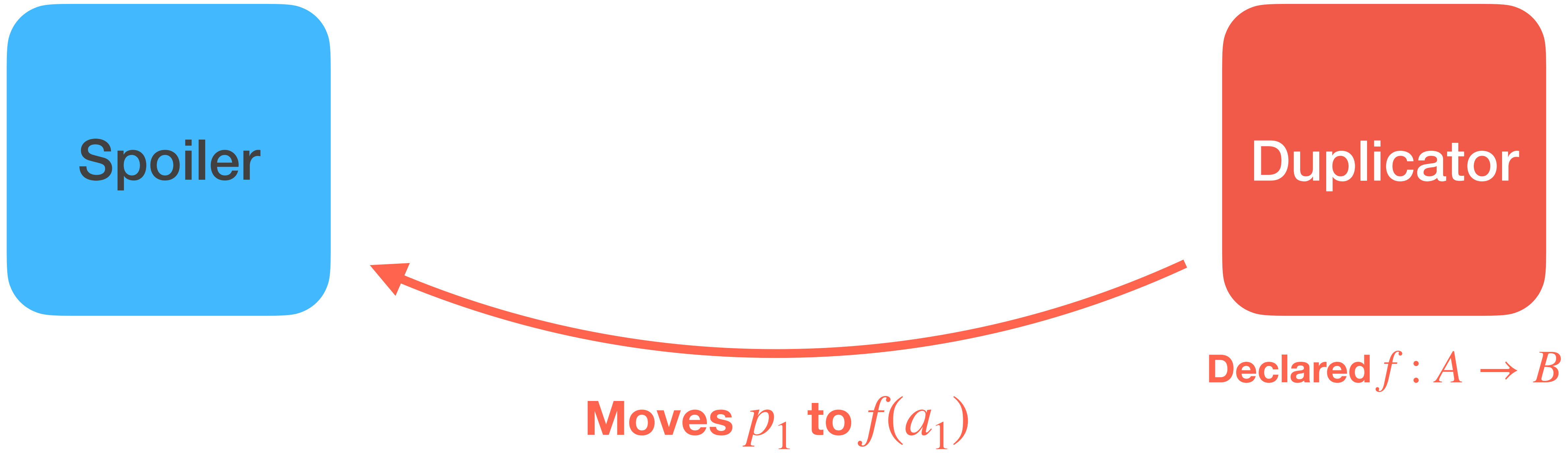
A "special" homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$[(p_1, a_1)] \quad \mapsto$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's "n-consistent" strategy for $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$

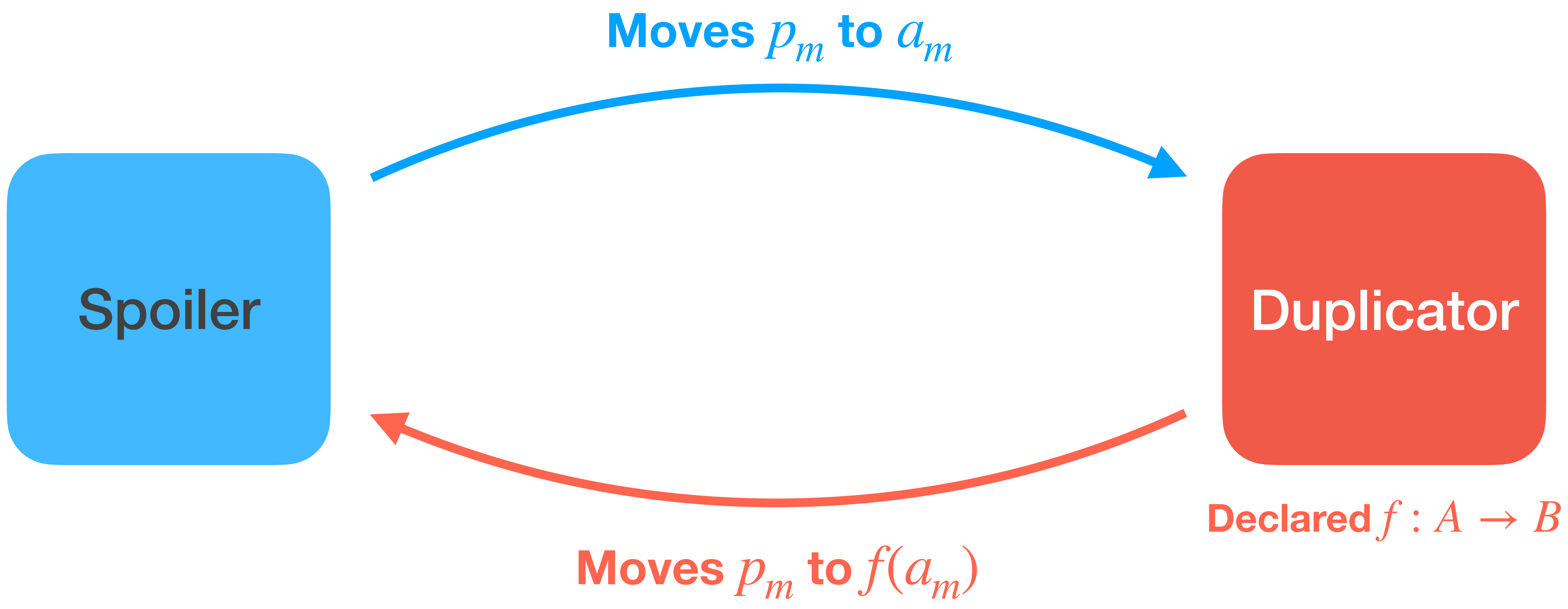
A "special" homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$



$$[(p_1, a_1)] \mapsto f(a_1)$$

Creating a new comonad from $\mathbb{P}_k$

Duplicator's "n-consistent" strategy for $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$



A "special" homomorphism $\mathbb{P}_k\mathscr{A} \to \mathscr{B}$

$$[(p_1, a_1)] \mapsto f(a_1)$$

$$\vdots$$

$$[(p_1, a_1), \ldots (p_m, a_m)] \mapsto f(a_m)$$

Game continues with Duplicator declaring a new f after Spoiler moves n pebbles (or earlier if Spoiler repeats a pebble).

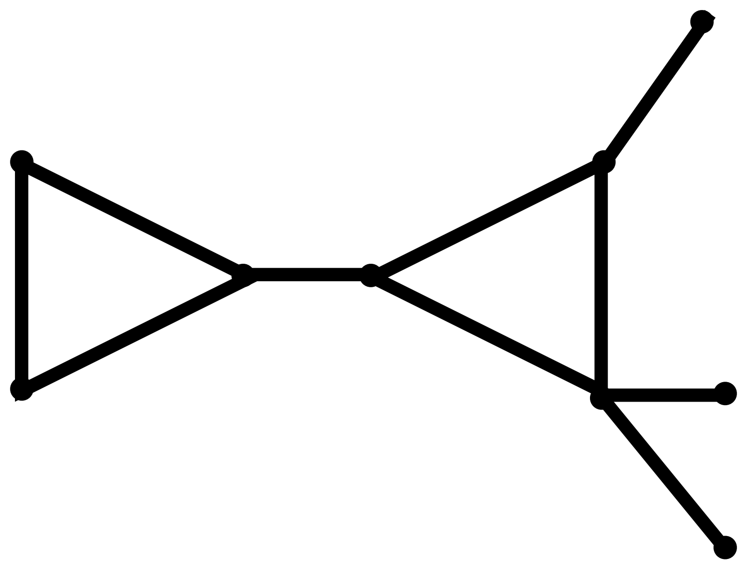
$\exists$ an equiv. rel. $\approx_n$ s.t. homomorphism $\mathbb{P}_k\mathscr{A}/\approx_n \to \mathscr{B} \iff$ n-consistent strategy for Duplicator in $\exists\text{Peb}_k(\mathscr{A}, \mathscr{B})$

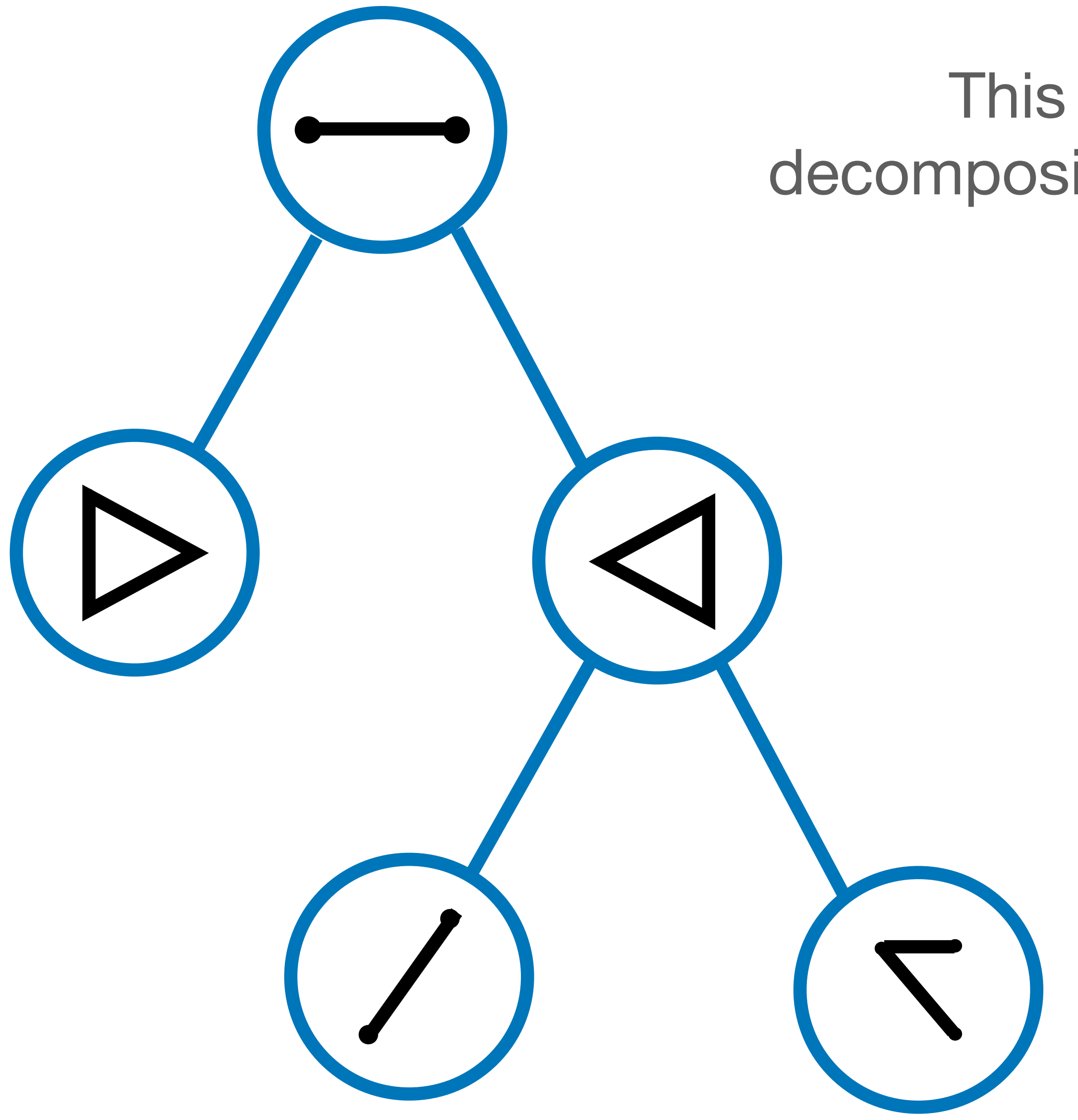
$\iff$ strategy for Duplicator in $+\text{Fun}^k_n(\mathscr{A}, \mathscr{B})$

Coalgebras are decompositions: revisited

Tree decompositions of a relational structure

Robertson & Seymour pioneered the study of taking a relational structure and studying its decompositions such as that below





This is a tree decomposition of width 2

Abramsky, Dawar & Wang 2017

There exists $\alpha : \mathscr{A} \to \mathbb{P}_k\mathscr{A}$ a coalgebra

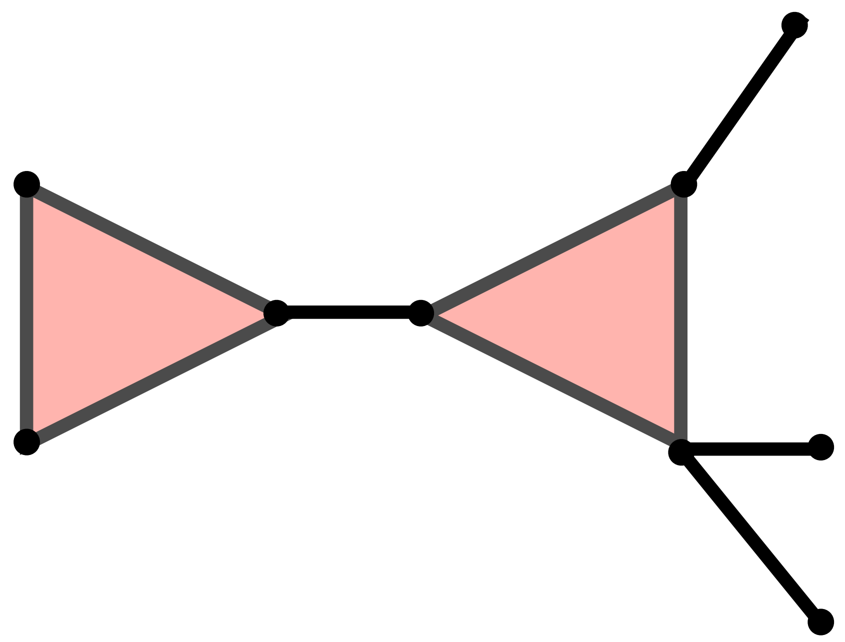
$\iff$ $\mathscr{A}$ has a tree decomposition of width k

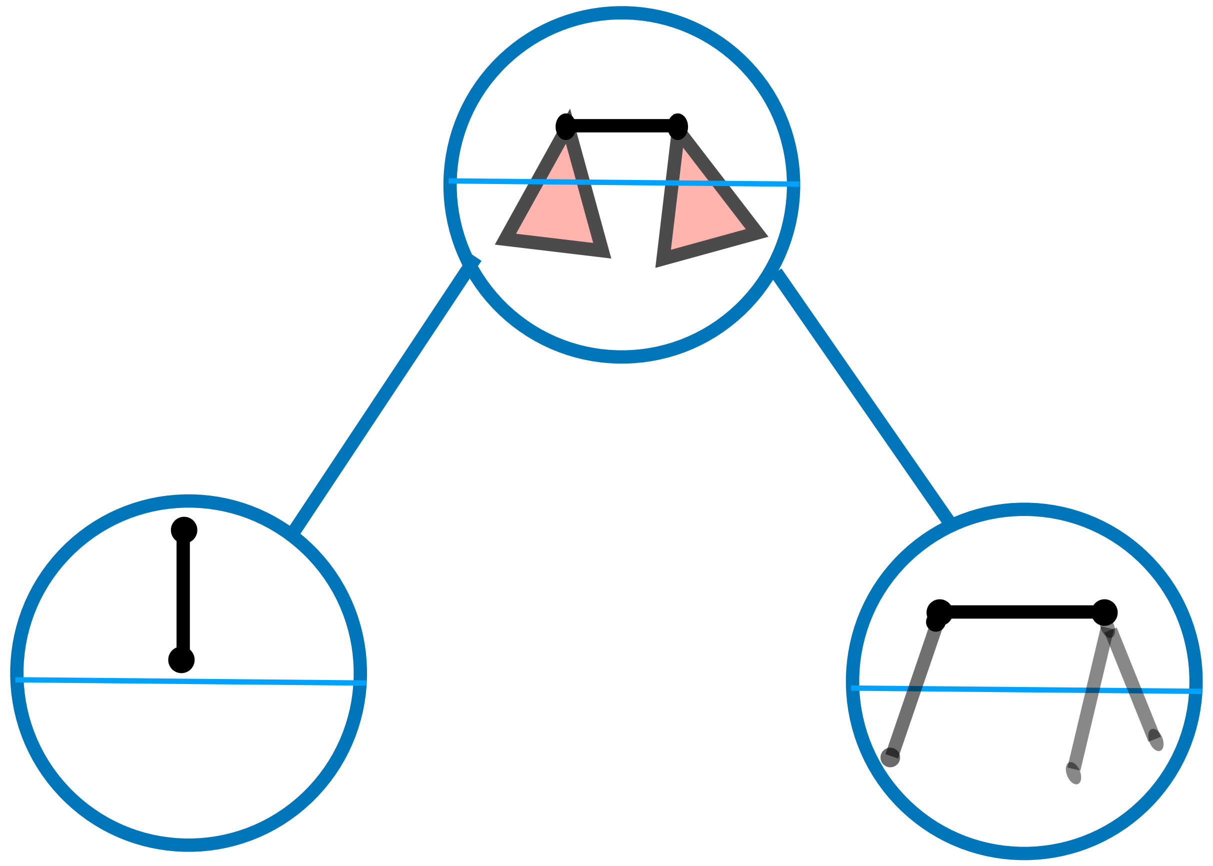
Coalgebras are decompositions: revisited

Ó C & Dawar, 2021

There exists $\alpha : \mathscr{A} \to \mathbb{G}_{n,k}\mathscr{A}$ a coalgebra

$\iff$ $\mathscr{A}$ has an extended tree decomposition of width k and arity n



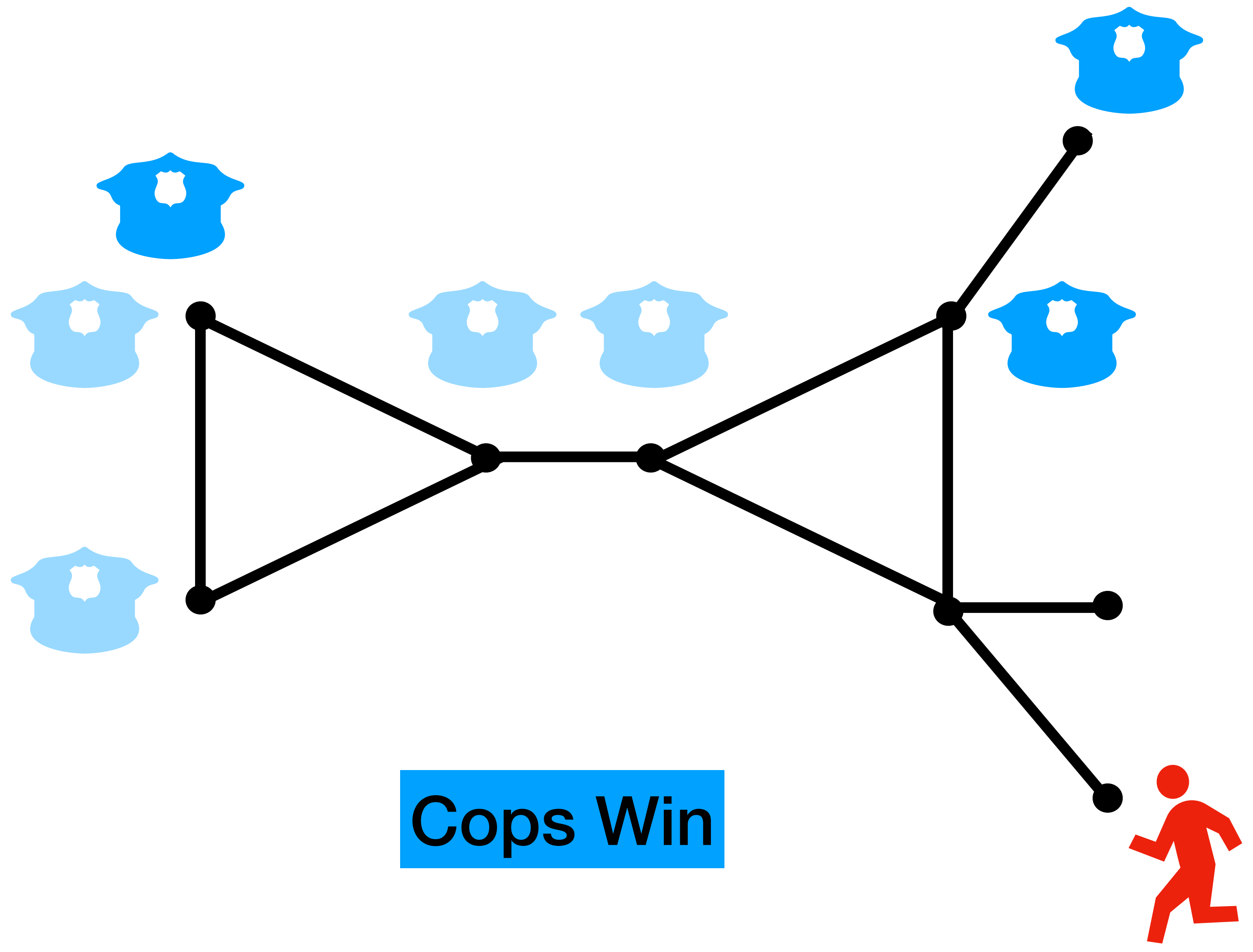


This is an extended tree decomposition of width 1 and arity 2

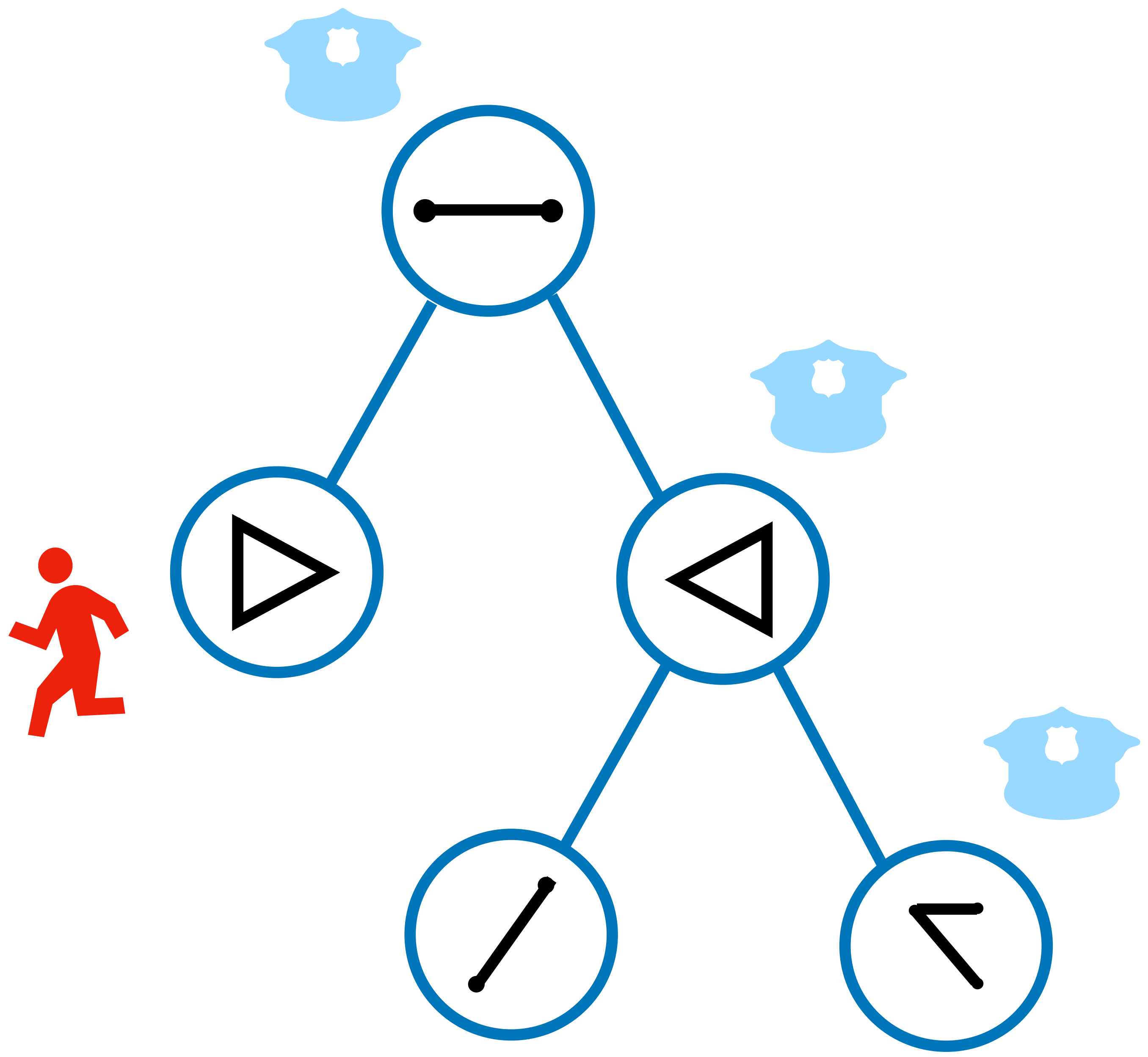
Coalgebras are decompositions: revisited

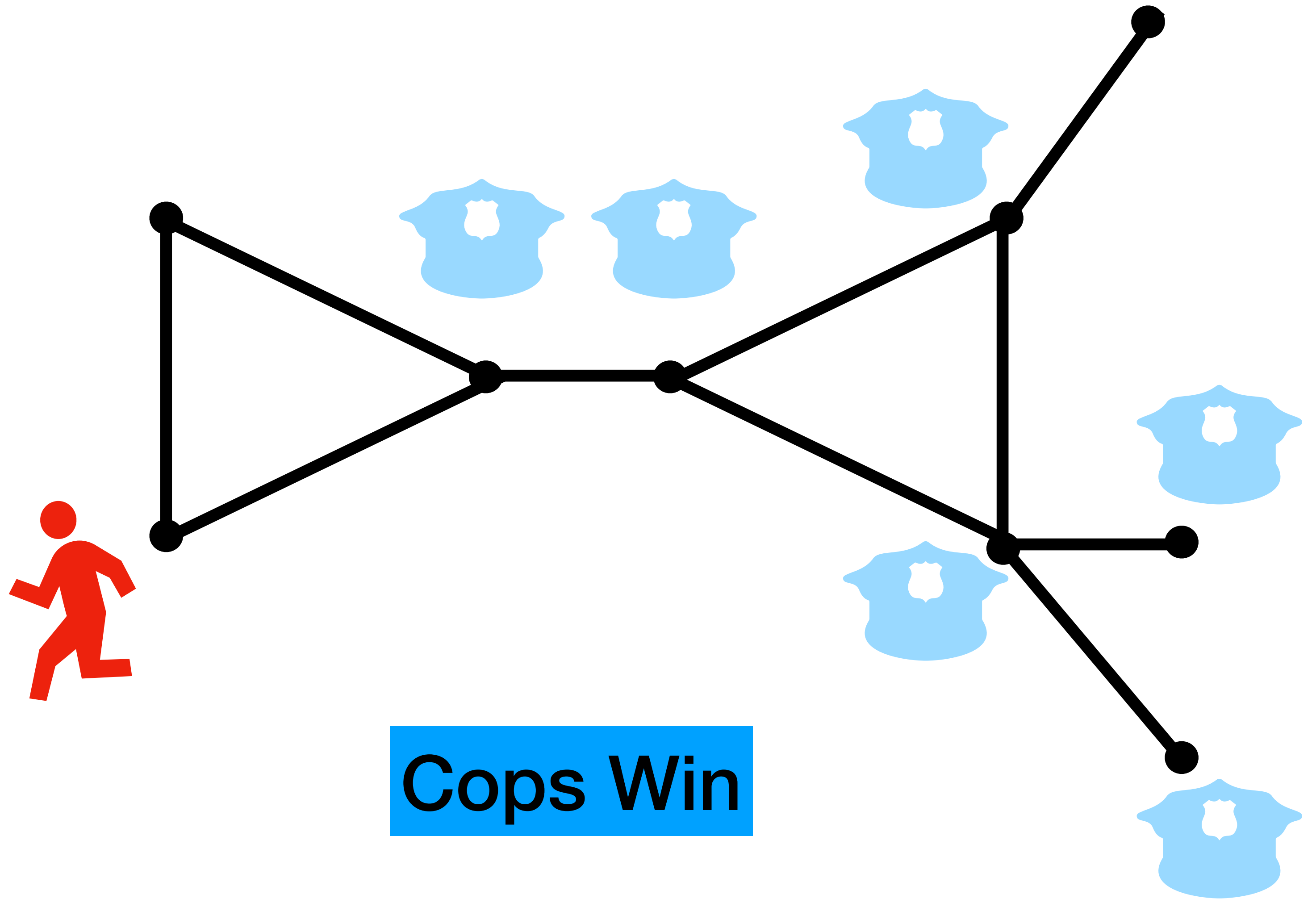
k cops & robber game on a graph

- k cops occupy k nodes of the graph, the robber occupies one node
- On each turn any number of cops can fly between any nodes of the graph but they must announce their moves to the robber ahead of time and once in the air they are removed from the board
- The robber responds by running (along edges of the graph) without passing through a niode occupied by a (stationary) cop.
- The cops in the air then land and if the robber is in on a node now occupied by a cop, he loses.
- The robber wins by evading capture indefinitely



Coalgebras are decompositions: revisited

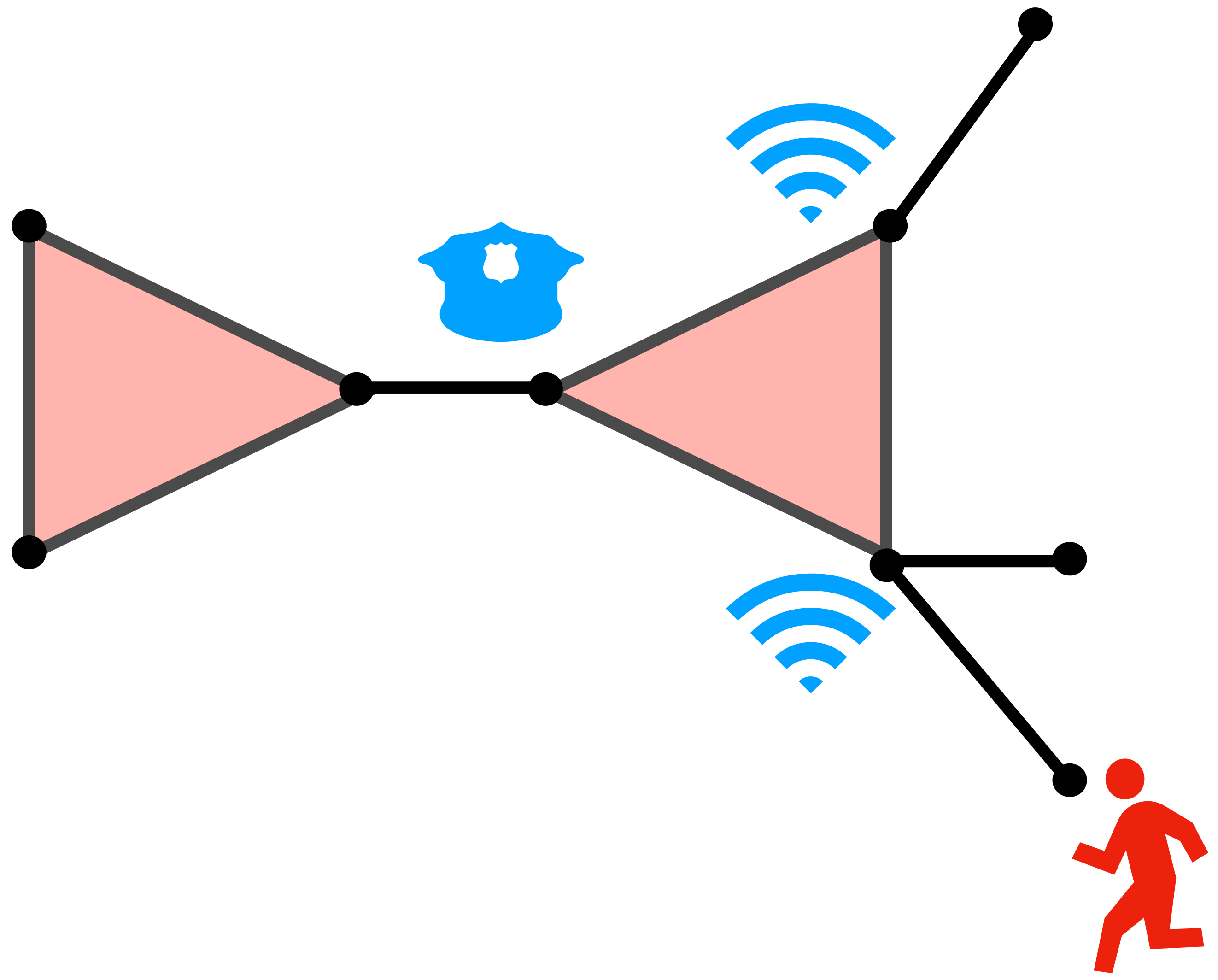




Coalgebras are decompositions: revisited

k cops, n-beacon & robber game on a hypergraph

- Similar to the cops and robbers game with two differences
- (a) Cops can now light any number of beacons on each turn
- (b) Robber can move through any two vertices connected by a hyperedge except if either vertex is occupied by a cop or the entire edge is filled with cops and beacons and there are at most n beacons.
- Cops still win if they can



Coalgebras are decompositions: revisited