Extending FO logic with topology

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joint with *Anuj Dawar and Yoàv Montacute*

September 2021

Comonads & Quantifiers

$$\mathbb{P}_k, \mathbb{E}_n, \mathbb{M}_n \quad \underset{\text{isom in } \mathbf{Kl}}{\longrightarrow} \quad \mathcal{L}^k(\#), \mathcal{L}_n(\#), \mathcal{M}_n(\#)$$

Comonads & Quantifiers

$$\mathbb{H}_{n,k} \quad \underset{\text{isom in } \mathbf{Kl}}{\longrightarrow} \quad \mathcal{L}^k(\mathbf{Q}_n)$$

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Comonads & Quantifiers

$\mathbb{H}_{n,k}$	$\xrightarrow[\text{isom in } \mathbf{Kl}]{}$	$\mathcal{L}^k(\mathbf{Q}_n)$
???	$\xrightarrow[\text{isom in } \mathbf{Kl}]{}$	$\mathcal{L}^k(\mathbf{LA})$
$\mathbb{P}_k, \mathbb{E}_n, \mathbb{M}_n$	$\xrightarrow[\text{isom in } \mathbf{Kl}]{}$	$\mathcal{L}^k(\#), \mathcal{L}_n(\#), \mathcal{M}_n(\#)$

Comonads & Quantifiers

$\mathbb{H}_{n,k}$	$\underset{\text{isom in } \mathbf{Kl}}{\longrightarrow}$	$\mathcal{L}^k(\mathbf{Q}_n)$
$???$	$\underset{\text{isom in } \mathbf{Kl}}{\longrightarrow}$	$\mathcal{L}^k$(Topological Quantifiers?)
$\mathbb{P}_k, \mathbb{E}_n, \mathbb{M}_n$	$\underset{\text{isom in } \mathbf{Kl}}{\longrightarrow}$	$\mathcal{L}^k(\#), \mathcal{L}_n(\#), \mathcal{M}_n(\#)$

Content

1. Why topology?
2. Generalised topology quantifiers
3. The topology of homsets

Part 1: Why topology?

The original **CFI construction** is a pair of transforms $X_{-}, \tilde{X}_{-}$ sending 3-regular graphs to 3-regular graphs.
For any k and $\mathcal{G}$ of large enough tree width, we have

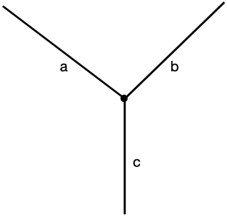
$$X_{\mathcal{G}} \equiv_{\mathcal{L}^k(\#)} \tilde{X}_{\mathcal{G}} \text{ but } X_{\mathcal{G}} \ncong \tilde{X}_{\mathcal{G}}$$

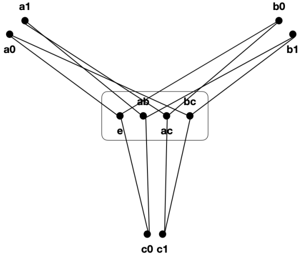
Rank and linear algebraic quantifiers distinguish them

$$X_{\mathcal{G}} \not\equiv_{\mathcal{L}^k(\mathbf{LA})} \tilde{X}_{\mathcal{G}}$$

Vertices in CFI construction

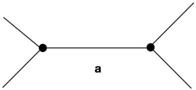
A vertex in $\mathcal{G}$ $\longrightarrow$ "Gadget" in $X_{\mathcal{G}}$



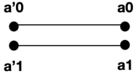


Edges in CFI construction

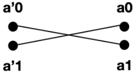
An edge a in $\mathcal{G}$



Untwisted edge in $X_{\mathcal{G}}$



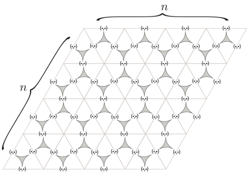
Twisted edge in $\tilde{X}_{\mathcal{G}}$



$\tilde{X}_{\mathcal{G}}$ contains exactly one "twisted" edge.

Why topology?

- $\mathcal{G}$ as triangulation of torus



- 2-regular CFI graphs distinguished by connectedness

Vertex in $\mathcal{G}$ $\longrightarrow$ Vertex in $X_{\mathcal{G}}$

- Grohe implies $L^k(\#)$ captures isom on any class of graphs of bounded genus.

Question 1 Can we devise a (tractably computable) topology-based equivalence relation $\equiv_{\mathcal{T}}$ which distinguishes $X_{\mathcal{G}}$ and $\tilde{X}_{\mathcal{G}}$?

Question 2 Is there a categorical semantics for any such equivalence relation?

Part 2: Topology quantifiers

Interpretation in logic

$\Psi^T(\mathbf{x}, \mathbf{y})$ a *type* T *interpretation* is a collection of $\mathcal{L}[\sigma]$ formulae in $\mathbf{x}$ and $\mathbf{y}$ which defines for any structure $\mathcal{A}$ and any asgmt $\mathbf{a}$ to variable $\mathbf{x}$ an object of type T which depends only on the truth values of the formulas in Ψ^T in $\mathcal{A}, \mathbf{a}$ as $\mathbf{y}$ varies over A^n

Examples

- $\Psi^{\mathbf{Set}} = \psi(\mathbf{x}, \mathbf{y})$
  interprets a **set** for any $\mathcal{A}, \mathbf{a}$
- $\Psi^{\tau} = \langle \psi_R(\mathbf{x}, y_1, \ldots y_{n_R}) \rangle_{R \in \tau}$
  interprets a τ-**structure** for any $\mathcal{A}, \mathbf{a}$
- $\Psi^{\mathbb{F}\mathbf{Mod}} = \langle \psi_{M_i}(\mathbf{x}, y_1, \ldots y_{2n}) \rangle_{i \in [r]}$
  interprets a **tuple of** r $n \times n$ **0-1 matrices over** $\mathbb{F}$ for any $\mathcal{A}, \mathbf{a}$

Quantifiers

A type T quantifier $\mathcal{Q}^T_{\mathcal{K}}$ for $L[\sigma]$ is described by an isom-closed class $\mathcal{K}$ of T objects and binds the $\mathbf{y}$ vairables of $\Psi^T(\mathbf{x}, \mathbf{y})$ an $L[\sigma]$-interpretation of type T.

$\mathcal{Q}^T_{\mathcal{K}}\mathbf{y}.\Psi^T(\mathbf{x}, \mathbf{y})$ is true on $\mathcal{A}, \mathbf{a}$ if and only is the corresponding T object is in $\mathcal{K}$

Examples

- $\Psi^{\mathbf{Set}} = \psi(\mathbf{x}, \mathbf{y})$
  quantifiers $\mathcal{Q}_{\mathcal{K}}^{\mathbf{Set}}$ are [1] counting quantifiers $\exists^{\leq t}$
- $\Psi^{\tau} = \langle \psi_R(\mathbf{x}, y_1, \ldots y_{n_R}) \rangle_{R \in \tau}$
  quantifiers $\mathcal{Q}_{\mathcal{K}}^{\tau}$ are generalized quantifiers of τ structures
- $\Psi^{\mathbb{F}\mathbf{Mod}} = \langle \psi_{M_i}(\mathbf{x}, y_1, \ldots y_{2n}) \rangle_{i \in [r]}$
  quantifiers $\mathcal{Q}_{\mathcal{K}}^{\mathbb{F}\mathbf{Mod}}$ are linear-algebraic quantifiers over $\mathbb{F}$

[1] arbitrary conjunctions and disjunctions of

Interpreting simplicial complexes

Abstract simplicial complex $\mathbf{\Delta} = (V, S)$ a pair of vertex set V and simplex set S, a downward closed subset of 2^V
Dimension of a simplex $s \in S$ is $|s| - 1$.

$$\Psi^{\mathbf{Simp}_n} = \langle \psi_m(\mathbf{x}, y_1, \ldots y_m) \rangle_{m \in [n+1]}$$

Interpretation of n-dimensional abstract simplicial complex from a σ-structure.

Topological quantifiers

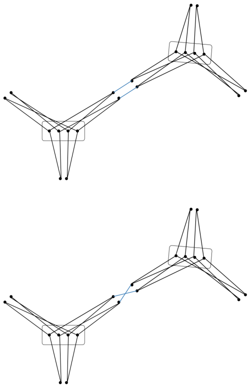
$$\mathcal{Q}_{\mathcal{K}}^{\mathbf{Simp}_n}$$

Which classes $\mathcal{K}$ should be allowed?

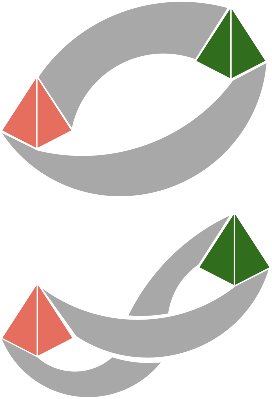
$\cong$-closed?	As hard as GI
$\simeq_h$-closed?	??
$\simeq_H$-closed?	Tractable

An attempt at distinguishing CFI graphs

CFI graphs



Interpreted complexes



Sadly this interpretation did not create complexes for $X_{\mathcal{G}}$ and $\tilde{X}_{\mathcal{G}}$ with different homology!

Open questions

- Can these complexes be distinguished by some other tractable topological property?
- Can we express the homology-based quantifiers $\mathcal{Q}_{\mathcal{K}}^{\mathbf{Simp}_n}$ in $\mathcal{L}^k(\#)$?
- Can we interpret another complex on $X_{\mathcal{G}}, \tilde{X}_{\mathcal{G}}$ which does distinguish them topologically?

Part 3: A Lovasz-type equivalence with topology

Lovasz-type results

$$\mathcal{A} \equiv_{\mathcal{L}} \mathcal{B} \iff \forall \mathcal{C} \in F \quad |\mathrm{hom}(\mathcal{C}, \mathcal{A})| = |\mathrm{hom}(\mathcal{C}, \mathcal{B})|$$

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Some known examples

$\mathcal{L}$	F
$\cong$	$\mathcal{R}_f(\sigma)$
$\mathcal{L}^k(\#)$	$\mathcal{W}_k$
$\mathcal{L}_n(\#)$	$\mathcal{T}_n$

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...and many more thanks to Tomas, Luca and Anuj!

Lovasz-type results

$$\mathcal{A} \equiv_{\mathcal{L}} \mathcal{B} \iff \forall \mathcal{C} \in F \quad \text{hom}(\mathcal{C}, \mathcal{A}) \overset{?}{\simeq} \text{hom}(\mathcal{C}, \mathcal{B})$$

Can we compare homsets in any way other than counting?

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Can we compare homsets in any way other than counting?

Yes! With topology . . .

Homomorphism complexes

For $\mathcal{A}, \mathcal{B}$ σ-structures, the *hom-complex* $\mathbf{Hom}(\mathcal{A}, \mathcal{B})$ is a simplicial complex[2] with vertices the homomorphisms $f : \mathcal{A} \to \mathcal{B}$ where $\{f_1, \ldots f_m\}$ form a simplex if for all $a_1, \ldots, a_n \in A$ and all $R \in \sigma$,

$$(a_1, \ldots, a_n) \in R^{\mathcal{A}} \implies \prod_i \{f_j(a_i)\}_{j \in [m]} \subseteq R^{\mathcal{B}}$$

[2] other mostly equivalent notions exist

Example

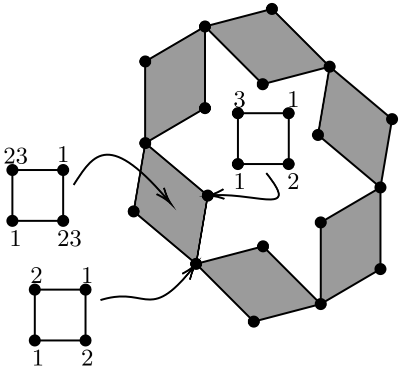

Figure: $\mathrm{Hom}(C_4, K_3)$

Further examples

See board!

$$\mathbf{Hom}(K_2, C_6) \cong \mathbf{Hom}(K_2, C_3 \cup C_3)$$

Hom-complexes and Lovász-type result

- The 1-skeleton of the hom-complex can be thought of as a simple undirected graph where the 0-dimensional cells are its vertices and the 1-dimensional cells are its edges.

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Theorem (Kozlov)

For every two graphs G and H, the isomorphism type of $\mathrm{Hom}(G, H)$ is determined by the isomorphism type of $\mathrm{Hom}^1(G, H)$.

Hom-complexes and Lovász-type result

Question.

Suppose A and B are finite (undirected) graphs such that

$$A \simeq^{\mathcal{C}} B \iff \forall F \in \mathcal{C}, Hom^1(F, A) \cong Hom^1(F, B).$$

For which class $\mathcal{C}$ the relation $\simeq^{\mathcal{C}}$ is isomorphism $\cong$?

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- While we ask these questions for isomorphism-equivalence and finite undirected graphs, we would like to generalise them to homotopy-equivalence and general finite σ-structures.

Preliminary results

- For finite, simple, undirected graphs

$$\mathcal{G} \simeq^{\mathbf{J_2}} \mathcal{H} \implies \mathcal{G} \equiv_{\mathcal{L}^2(\#)} \mathcal{H}$$

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- For finite, simple, undirected graphs

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- However ...
- For graphs with loops

$$\mathcal{G} \simeq^{\mathbf{J_2}} \mathcal{H} \not\Longrightarrow \mathcal{G} \equiv_{\mathcal{L}^2(\#)} \mathcal{H}$$

- For $k > 2$ and simple graphs

$$\mathcal{G} \simeq^{\mathbf{J_k}} \mathcal{H} \not\Longrightarrow \mathcal{G} \equiv_{\mathcal{L}^k(\#)} \mathcal{H}$$

Open questions and new directions

- Treat $\mathcal{C}$ as a category and define $\mathcal{A} \simeq^{\mathcal{C}}_{\mathbf{nat}} \mathcal{B}$ to be witnessed by natural isomorphisms

$$Hom(-, \mathcal{A}) \cong Hom(-, \mathcal{B})$$

- Can we find a game for which it is sensible to talk about the Hom-complex $\mathrm{Hom}(\mathbb{C}_k\mathcal{A}, \mathcal{B})$ for some $\mathbb{C}_k$ in the same way we talk about hom$(\mathbb{T}_k\mathcal{A}, \mathcal{B})$ in the context of the existential k-pebble game?
- Can we find a Lovasz-type result based on Hom complexes? (e.g. proving $\simeq^{\mathcal{C}}$ equivalent to $\equiv_{\mathcal{L}}$ for some logic)