Second Year Report: *Game comonads for linear-algebraic resources in finite model theory*

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July 31, 2020

1 Introduction

As discussed in detail in my first year report[10], my research sits at an interesting crossroads of finite model theory, descriptive complexity theory and category theory. In particular, I focus on exploring extensions of the notion of *game comonad* introduced by Abramsky, Dawar and Wang's seminal paper on the *pebbling comonad*[2]. This work used a comonad $\mathbb{P}_k$ to present connections between the existential k-pebble game, k Weisfeiler-Lehman equivalence and tree decompositions of width $< k$ in a very elegant way. My work has sought to understand how similar comonadic constructions can be used to capture the semantics and relationships between model-comparison game which are important to the descriptive complexity and finite model theory communities. In particular, I would like to expand this framework to cover games for the linear-algebraic logics of Dawar and Holm[8] and Dawar, Grädel and Pakusa [4] which are currently the best candidates for logics which capture PTIME. I have so far found this area to be very fruitful for research and have already had success with constructing a new game comonad for Hella's n-bijective k-pebble game, giving a comonadic semantics for generalised quantifiers. This report details how I plan to build on this success towards the completion of my thesis.

2 Progress

In my First Year Report [10], I presented two possible paths towards a thesis in the form of the dependency diagrams in Figures 1 & 2. Throughout my second year, I have used these plans to guide my research. In the course of the year, I've made progress in a number of the areas listed and I've been able to narrow down these plans into one coherent project which will become my thesis.

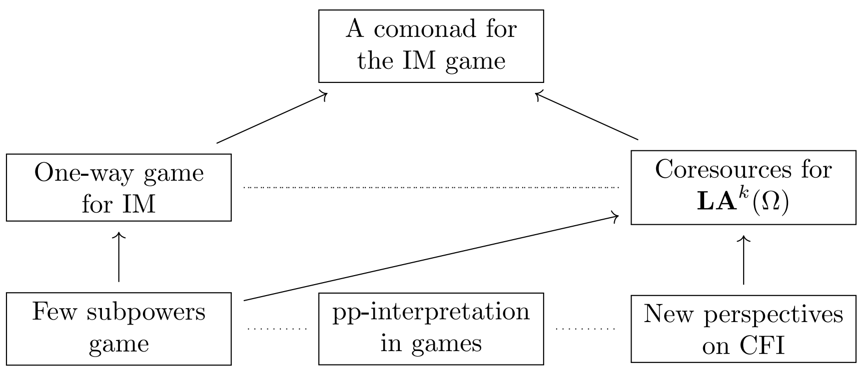

Figure 1: Path A

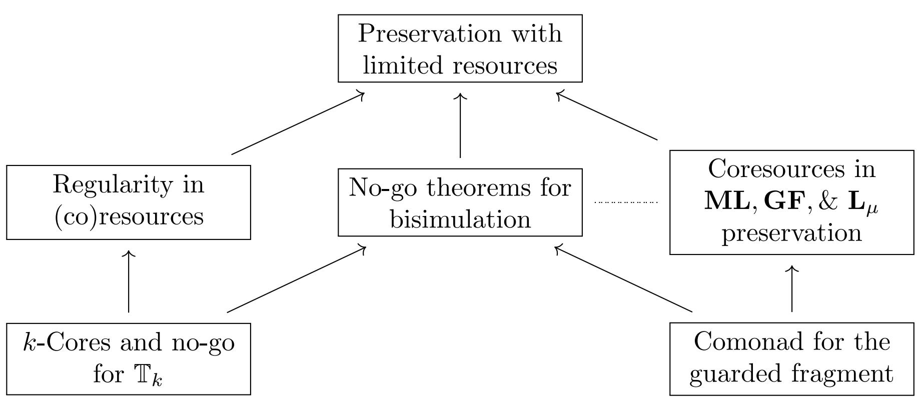

Figure 2: Path B

2.1 Deciding on Path A

The wishlist presented in the figures above was, in the words of my supervisor this time last year *"enough for an entire programme of research rather than a single thesis"*. Thus a crucial element of the progress I have made in my research this year has been in focussing this outline into a single thesis-sized project. To this end, I have pruned most of the items in Path B (with the exception of "k-cores and no-go theorems") and progressed with Path A as I show later in this section and in Section 3.

My main reasons for this are as follows:

- Path A (plus the work on k-cores) is a more coherent collection of topics with a clear motivation. Namely understanding how generalised linear algebraic resources added to first-order logic can be understood in the categorical framework of Abramsky et al.
- I have made substantial progress on our work so far in Path A, as I detail below.
- Various parts of Path B are now being covered by other members of Abramsky and Dawar's *Resources & Coresources* programme.

2.2 Progress made

The main success I've had in following this plan has been the discovery of a new game comonad in the style of Abramsky, Dawar and Wang's pebbling comonad [2]. This work, detailed in the pre-print [11] and submitted to CSL 2021, took Hella's k-pebble n-bijective game [7] which captures equivalence of structures in k-variable infinitary logic extended by all n-ary Lindström quantifiers [9], derived from it an appropriate one way game and showed how a game-theoretic process of translating strategies from this game to the one way k pebble game could be used to "lift" Abramsky, Dawar and Wang's $\mathbb{P}_k$ to a new game comonad $\mathbb{G}_{n,k}$ for Hella's game. While this didn't appear in the plan of my first year report, it emerged from work I was doing on the IM-game and has provided a template for how I could tackle this game in the future. This game, introduced by Dawar and Holm, is, like Hella's game, an inherently two way game, thus the experience of constructing the $\mathbb{G}_{n,k}$ comonad has opened up new avenues to try in the search for a comonad for the IM-game as I will describe in Section 4.

In addition to this unexpected progress, I've also made progress towards some of the items on my plans from last year. I give details of these below:

- **k-cores & no-go theorems**: In my initial work in defining and studying k-cores, I posed some open questions about whether or not unique k-cores could exist in the Kleisli category of $\mathbb{P}_k$. In trying to answer these, I related monomorphisms and epimorphisms in $\mathcal{K}(\mathbb{P}_k)$ with preservation of certain logical formulas between structures. In my work on $\mathbb{G}_{n,k}$ I have found a much more general and natural version of this relationship which seems like a promising starting point to answering my previous questions about k-cores. Prof. Dawar and I are currently assembling a revised set of notes on this topic which we hope will lead to another publication.

- **pp-interpretation in games**: In general, primitive-positive interpretations of a signature τ in a signature σ are functors $\Phi : \mathcal{R}(\sigma) \to \mathcal{R}(\tau)$ which can be defined with primitive-positive first order formulas. This is something which isn't treated abstractly in current game comonad literature despite its seemingly categorical nature. As generalised quantifiers are essentially built on interpretations, our work on $\mathbb{G}_{n,k}$ has opened up questions about how exacltly interpretations arise in the comonad framework. Further to this, I have had productive conversations and collaborations this year with Martin Hyland and Glynn Wynskel from the game semantics community. From this I have written a short set of notes on interpretations which I am confident can be extended into a contribution to my thesis.
- **New perspectives on CFI**: My work on this theme has uncovered an interesting new direction in understanding algebraic logical resources and a successful research proposal to the Alan Turing Institute's *PhD Enrichment Programme*. In exploring the CFI construction this year, I helped to organise a reading group on the construction in the lab and had interesting conversations with Alexandra Silva and Jamie Tucker-Foltz on the topological properties of this elusive but ubiquitous construction. My work next year at the Turing Institute will involve working with both logicians and members of the topological data analysis community to try and develop a clearer understanding of the expressive power of adding the ability to compute homology of logically-interpreted simplicial complexes to k-variable infinitary counting logic.

Collectively this progress lays a solid foundation for a thesis on algebraic resources and coresources in finite model theory. In Section 3 I will lay out how these pieces might fit together in my thesis and in Section 4 I will describe the concrete steps I plan to take to get from here to there.

3 Thesis Outline

In this section, I provide a proposed outline for my thesis, broken down into chapters. For each chapter I have given a description of what material I intend to include and a short update on my progress towards completing this chapter.

3.1 Outline of Thesis Chapters

1. **Background**
   This section would be a mostly standard recap of model comparison games

and related results and constructions and their importance in finite model theory, descriptive complexity and algorithms. This will refine the literature review from [10] and the background section of [11]
This chapter will adapt background material which I have already written.

2. **A review of game comonads**
As mine will be one of the first theses entirely focused on this new categorical approach to resources and coresources in finite model theory, I want to give a comprehensive and up-to-date survey of the latest results and directions in this newly-emerging field. This will help to firmly situate my contributions within this broader programme.
This chapter is currently in the form of an ongoing set of notes which will be expanded to remain up-to-date.

3. **Generalised quantifiers in game comonads**
This chapter will describe my work (with Anuj Dawar) on generalised quantifiers as a logical resource. This will include the main results from [11] and extensions on these. It will show how a very powerful type of logical resource (one which in a way subsumes the more refined linear-algebraic quantifiers) can be treated in the game comonads framework.
The material for this chapter is complete. [11]

4. **Categorical treatment of interpretation in generalised quantifiers**
The categorical setting for most of finite model theory is the category $\mathcal{R}(\sigma)$ of relational structures and homomorphisms (over some *fixed* signature σ). All of the work on game comonads (including that in the previous chapter) has been done in this setting. However, generalised quantifiers by nature involve interpretation between signatures which don't yet have category-theoretic treatment in the game comonads literature.
This chapter will provide this treatment starting with the categories Σ_L of signatures and L-interpretations (for different logics L) and studying $\mathcal{R}(\sigma)$ as a fibre over an element of this category. Understanding how this category interacts with the $\mathbb{G}_{n,k}$ comonad will be an essential part of this chapter and with this done it is hoped that we can relate different types of interpretation (for example interpretation into the **Vect**) to different families of generalised quantifiers.
This chapter is currently work in progress as described in Section 4

5. **Logics extended by linear-algebraic quantifiers**
This chapter will review logics which extend fixed-point logic with counting by adding certain linear-algebraic quantifiers and the model-comparison games which are known for these logics. Such logics were introduced to add enough expressive power to separate the structures of the Cai-Furer-Immerman construction [3] while still remaining decidable in polynomial-time. Existing examples include rank logic (introduced by Dawar and Holm [5]) and linear-algebraic logic (introduced by Dawar, Grädel and

Pakusa [4]) and their respective games, the matrix-equivalence game and the invertible-maps game. I also intend to introduce an extension of infinitary counting logic by a quantifier for expressing the homology groups of certain interpretable simplicial complexes (the computation of which is also linear algebra). Answering the question of whether or not such a logic can express the CFI query would be important for understanding the power of these linear-algebraic resources.
The work for this chapter is planned for later this year and the section on homology will be completed as part of a PhD Enrichment scheme at the Alan Turing Institute.

6. **New (co)monads for linear-algebraic (co)resources**
This chapter will show how logical resources from the previous chapter can be given a categorical semantics by building new game comonads using the techniques from Chapters 3 & 4. As demonstrated in [11], this will lead to new games which capture different classes of linear-algebraic quantifiers and new structure decompositions which prove control a relational structures relationship with its "linear-algebraic" information, in the same way that tree-decompositions control a structures relationship to its local information. Explaining the theory of how these (co)monads capture linear-algebraic resources in finite model theory and showing how they interact with each other will be the central contribution of this thesis and will bring in material from all the previous chapters.
This work is planned and the exact content will be subject to change depending on the work towards the previous chapters.

4 Timetable of Future Work

Over the next year I'll be building on the success I've had this year and last in finding new ways to use category theory to understand model-comparison games and resources in finite model theory. Figure 3 presents my plan for progressing from the results of the last two years towards the thesis outlined in Section 3. Below I break down the constituent parts of this plan further and give approximate timelines for finishing each component of the work.

4.1 Work Completed

This work is completed and will be worked into the final thesis as indicated in Section 3

Results about injective/surjective strategies This work was completed in my first year in a set of notes on *k-Cores in the $\mathbb{P}_k$ comonad*. This work proved logical classifications of the *branch-injective/surjective* strategies mentioned by Abramsky et al. in [2] which were precursors of the more systematic classifications presented in [11]. Following Hell and Nešetřil[6] I also defined

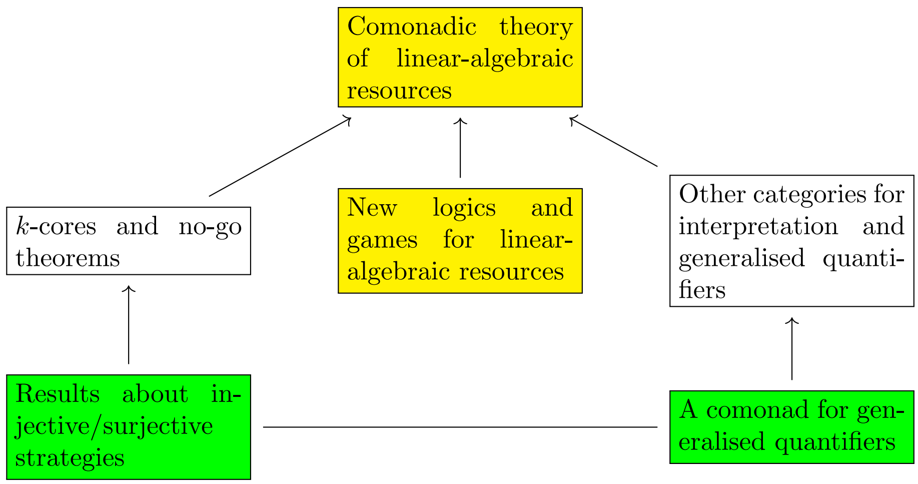

Figure 3: Dependency diagram for work towards thesis completion. Green boxes indicate completed work. Clear boxes indicate work currently in progress. Yellow boxes indicate work which is planned to start from January 2021.

a notion of a k-core for the Kleisli category of a game comonad in this work. Some open questions about this laid the foundations for my current work on *k-cores and no-go theorems*. Because of some interesting connections between this work and the work of Abramsky, Soares de Barbosa, de Silva and Zapata on the Quantum Monad[1], I presented this work at the workshop on *Contextuality as a resource in quantum computation* in Oxford in July 2019.

A comonad for generalised quantifiers This work has been a major achievement of my research this year. It establishes a game comonad for Hella's n-bijective k-pebble game. Along the way we found:

- a new way to construct new game comonads from old ones using a notion of a translation between strategies
- an interesting collection of 8 related types of map in the Kleisli category of this new comonad which correspond with 8 variants of Hella's game and 8 distinct classes of generalised quantifiers
- a new generalisation of tree decomposition (corresponding to the coalgebras of this comonad) which captures when a structure is easily decomposable w.r.t. k-variable logic extend by generalised quantifiers up to arity n

There are many interesting open questions which come from this work. Of particular interest to my intended programme of reearch is how we can use this comonad to identify generalised quantifiers whose classes are closed not just under isomorphism of relational structures but isomorphism in some other category such as **Vect**. This is the foundation of my intended work further work on *Interpretation and generalised quantifiers*.

4.2 Work in Progress

This is ongoing work which has been planned for the second half of 2020. The intention is to finish these pieces of work by the end of the year, potentially submitting some elements for publication.

k-cores and no-go theorems This work involves extending my notes on k-cores from first year using new insights gained from my paper with Anuj this year and a set of notes Anuj has written about no-go therems for game comonads (which show that infinite comonads are necessary for certain logical resources). We aim to answer the following questions:

- Are there unique k-cores (or n, k-cores) in the Kleisli categories of $\mathbb{P}_k$ (or $\mathbb{G}_{n,k}$)?
- Can the existence of k-cores be characterised by a categorical condition on the Kleisli category? (Regularity appeared to be a strong contender in the initial work)
- Does the non-existence of unique cores imply (and is it implied by) a no-go theorem?
- Can these k-cores be related to the k-cores of Rossman's proof of the finite homomorphism preservation theorem?[12]
- If so, what can the (non-)existence of k-cores tell us about *equi-resource*[1] versions of the homomorphism preservation theorem?

It is hoped that this work can be written up as a paper to be submitted for publication. The results in it are likely to be relevant to the (co)monads we aim to find for linear-algebraic resources.

Other categories for interpretation and generalised quantifiers In our work on generalised quantifiers[11], it was noted that the class of quantifiers that are governed by a particular variant of Hella's game is determined by the kind of *functions* which Duplicator is allowed to play in the game. For example, unrestricted functions correspond to homomorphism-closed quantifiers and bijections correspond to isomorphism-closed quantifiers. In the pursuit of linear-algebraic quantifiers, we would like to show that Holm's IM-game[8] fits into some generalisation of this framework. To do this we want to answer the following questions:

[1] for example equirank or equivariable

- Duplicators responses in variants of Hella's game are morphisms in **Set** between the underlying sets of the relational structures involved. Is there a way to replace this forgetful functor from $\mathcal{R}(\sigma)$ with another functor?
- We have dealt with generalised quantifiers entirely in $\mathcal{R}(\sigma)$ up to this point. However, by their nature generalised quantifiers involve interpretations *between* signatures. Can we describe the categorical structure of these interpretations and if so can we talk about interpretations into other categories, for example **Vect**.

4.3 Work to Come

This is work which I intend to do from the beginning of 2021 until the completion of my thesis. Some of it relies on work which is currently in progress and so the exact details may change.

New logics and games for linear-algebraic resources In the last two years I have made a number of attempts at constructing a one-way version of the IM game and along the way have found some interesting candidates and some things which appear to describe other extensions of k variable logic with linear algebraic resources. The first part of this work will involve collecting these attempts and proving correspondences between these games and logics with linear algebraic resources (where these exist).
The second part of this work will involve introducing a totally new linear-algebraic logic inspired by my research this year on the topological nature of the CFI construction. This logic will add a quantifier for computing cohomology of interpreted simplices over various fields. I aim to answer whether such a logic can describe the CFI query and to develop a game for this logic. This work was proposed and accepted for the Alan Turing Institute PhD Enrichment Scheme whihc starts in January 2021. I'll use this as an opportunity to collaborate with members of the topological data analysis community who have knowledge of topological queries and interest in complexity results about them.

Comonadic theory of linear-algebraic resources This is the ultimate aim of my research project and will involve:

- Finding (co)monads which capture all (or some) of the existing and newly introduced linear algebraic model comparison games
- Developing new structure decompositions which arise from the (co)algebras of these constructions
- Showing that some (or all) of these constructions can be derived from the comonad for generalised quantifiers in an interesting way.

This work will likely coincide with the Turing Institute project and will continue afterwards.

5 Concluding Remarks

In this report I have demonstrated that I have been able to make significant progress on the research proposed in my first year report, including producing my first pre-print and submitted paper and that I have a clear path from here towards a compelling thesis. I hope the examiners of this report agree with this assessment and share my excitement for the work to come in this project.

Finally, it would be remiss of me to conclude this report without mentioning the exceptional circumstances that have overshadowed my research (and that of all of my fellow PhD students and colleagues in the Computer Lab) this year. The ongoing global pandemic has burdened many with unprecedented disruption, anxiety and loss. Much less seriously, the loss of the resources, in-person collaboration and sense of community of the William Gates Building has made research much more difficult for everyone (even theorists!). As a result, I owe an enormous amount of the progress I've made this year to the continued patience and support of my supervisor, Anuj Dawar, to the extraordinary work of Lise and Marketa and all of the administrators in the Lab and to the love and support of my family and friends. I'd like to thank all of these people and anyone else who has made it possible for me to continue working at this difficult time.

References

[1] Abramsky, S., Barbosa, R. S., de Silva, N., and Zapata, O. The Quantum Monad on Relational Structures. In *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017)* (Dagstuhl, Germany, 2017), K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin, Eds., vol. 83 of *Leibniz International Proceedings in Informatics (LIPIcs)*, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, pp. 35:1–35:19.

[2] Abramsky, S., Dawar, A., and Wang, P. The pebbling comonad in finite model theory. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (June 2017), pp. 1–12.

[3] Cai, J.-Y., Fürer, M., and Immerman, N. An optimal lower bound on the number of variables for graph identification. *Combinatorica 12*, 4 (Dec 1992), 389–410.

[4] Dawar, A., Grädel, E., and Pakusa, W. Approximations of isomorphism and logics with linear-algebraic operators. In *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece* (2019), vol. 132 of *LIPIcs*, pp. 112:1–112:14.

[5] Dawar, A., Grohe, M., Holm, B., and Laubner, B. Logics with rank operators. In *Proceedings of the 2009 24th Annual IEEE Symposium

on Logic In Computer Science (Washington, DC, USA, 2009), LICS '09, IEEE Computer Society, pp. 113–122.

[6] Hell, P., and Nešetřil, J. The core of a graph. *Discrete Mathematics 109*, 1 (1992), 117 – 126.

[7] Hella, L. Definability hierarchies of generalized quantifiers. *Annals of Pure and Applied Logic 43*, 3 (1989), 235 – 271.

[8] Holm, B. *Descriptive complexity of linear algebra.* PhD thesis, University of Cambridge, UK, 2011.

[9] Lindström, P. First order predicate logic with generalized quantifiers. *Theoria 32*, 3 (1966), 186–195.

[10] Ó Conghaile, A. First year report: Game comonads, descriptive complexity & finite model theory, 2019.

[11] Ó Conghaile, A., and Dawar, A. Game comonads & generalised quantifiers, 2020.

[12] Rossman, B. Homomorphism preservation theorems. *J. ACM 55*, 3 (Aug. 2008), 15:1–15:53.